

UNIQUE EQUILIBRIUM STATES FOR GEODESIC FLOWS IN NONPOSITIVE CURVATURE

K. BURNS, V. CLIMENHAGA, T. FISHER, AND D. J. THOMPSON

ABSTRACT. We study geodesic flows on compact rank 1 manifolds and prove that sufficiently regular potential functions have unique equilibrium states if the singular set does not carry full pressure. In dimension 2, this proves uniqueness for scalar multiples of the geometric potential on the interval $(-\infty, 1)$, and this is an optimal result. In higher dimensions, we obtain the same result on a neighborhood of 0, and give examples where uniqueness holds on all of \mathbb{R} . For general potential functions φ , we prove that the pressure gap holds whenever φ is locally constant on a neighborhood of the singular set, which allows us to give examples for which uniqueness holds on a C^0 -open and dense set of Hölder potentials.

1. INTRODUCTION

We study uniqueness of equilibrium states for the geodesic flow over a compact rank 1 manifold with nonpositive sectional curvature. In negative curvature, geodesic flow is Anosov and every Hölder potential has a unique equilibrium state. In nonpositive curvature, the flow is nonuniformly hyperbolic and may have phase transitions; the challenge is to exhibit a class of potential functions where uniqueness holds.

The first major result in this direction was Knieper's proof of uniqueness of the measure of maximal entropy using Patterson–Sullivan measures [16]. We use different techniques, inspired by Bowen's criteria to show uniqueness of equilibrium states [3]. This approach has been generalized by the second and fourth named authors, giving uniqueness of equilibrium states under non-uniform versions of Bowen's hypotheses [6]. We give conditions under which these techniques can be applied to geodesic flows on rank 1 manifolds, and demonstrate that these conditions are satisfied for a large class of potential functions.

Date: April 3, 2017.

2010 Mathematics Subject Classification. 37D35, 37D40, 37C40. 37D25.

Key words and phrases. Equilibrium states, geodesic flow, topological pressure.

V.C. is supported by NSF grants DMS-1362838 and DMS-1554794. T.F. is supported by Simons Foundation grant # 239708. D.T. is supported by NSF grant DMS-1461163.

Throughout the paper, $M = (M^n, g)$ will be a closed connected C^∞ Riemannian manifold with nonpositive sectional curvature and dimension n , and $\mathcal{F} = (f_t)_{t \in \mathbb{R}}$ will denote the geodesic flow on the unit tangent bundle T^1M . There are two continuous invariant subbundles E^s and E^u of TT^1M , each of dimension $n - 1$, which are orthogonal to the flow direction E^c in the natural Sasaki metric; these can be interpreted as normal vector fields to the stable and unstable horospheres. If the curvature is strictly negative, \mathcal{F} is Anosov and $TT^1M = E^s \oplus E^c \oplus E^u$ is the Anosov splitting.

In nonpositive curvature, E_v^s and E_v^u may have nontrivial intersection; the *rank* of a vector $v \in T^1M$ is $1 + \dim(E_v^s \cap E_v^u)$. Equivalently, the rank is the dimension of the space of parallel Jacobi vector fields for the geodesic through v . The *rank of M* is the minimum rank over all vectors in T^1M . We assume that M has rank 1. For a rank 1 manifold, the *regular set*, denoted Reg , is the set of $v \in T^1M$ with rank 1. The *singular set*, denoted Sing , is the set of vectors whose rank is larger than 1. If Sing is empty, then the geodesic flow is Anosov; this includes the negative curvature case. The case when Sing is nonempty is a prime example of nonuniform hyperbolicity.

We study uniqueness of equilibrium states for the geodesic flow \mathcal{F} . An *equilibrium state* for a continuous function $\varphi: T^1M \rightarrow \mathbb{R}$, which we call a *potential function*, is an invariant Borel probability measure that maximizes the free energy $h_\mu(\mathcal{F}) + \int \varphi d\mu$, where $h_\mu(\mathcal{F})$ is the measure-theoretic entropy with respect to the geodesic flow. This maximum is denoted by $P(\varphi)$ and is called the *topological pressure* of φ with respect to the geodesic flow \mathcal{F} . In the case when $\varphi = 0$, the topological pressure is the topological entropy $h_{\text{top}}(\mathcal{F})$. Since \mathcal{F} is entropy expansive, equilibrium states exist for any continuous function, but uniqueness is a subtle question beyond the uniformly hyperbolic setting.

The *geometric potential* $\varphi^u(v) = -\lim_{t \rightarrow 0} \frac{1}{t} \log \det(df_t|_{E_v^u})$ and its scalar multiples $q\varphi^u$ ($q \in \mathbb{R}$) are of particular interest. When $q = 1$, the Liouville measure μ_L is an equilibrium state for φ^u ; in the Anosov setting, it is the unique equilibrium state. When $q = 0$, equilibrium states for $q\varphi^u$ are measures of maximal entropy; uniqueness of the measure of maximal entropy in rank 1 was proved by Knieper [16]. In the case of surfaces without focal points, this result has been established recently using different methods by Gelfert and Ruggiero [12]. When M is a rank 1 surface, the family $q\varphi^u$ contains geometric information about the spectrum of the maximum Lyapunov exponent [5].

We now state our main theorems. Let $P(\text{Sing}, \varphi)$ denote the topological pressure of the potential $\varphi|_{\text{Sing}}$ with respect to the geodesic flow restricted to the singular set (setting $P(\text{Sing}, \varphi) = -\infty$ if $\text{Sing} = \emptyset$).

Theorem A. *Let \mathcal{F} be the geodesic flow over a closed rank 1 manifold M and let $\varphi: T^1M \rightarrow \mathbb{R}$ be $\varphi = q\varphi^u$ or be Hölder continuous. If $P(\text{Sing}, \varphi) < P(\varphi)$, then φ has a unique equilibrium state μ . This measure satisfies $\mu(\text{Reg}) = 1$, is fully supported, and is the weak* limit of weighted regular periodic orbits (see Section 2.3).*

The hypothesis $P(\text{Sing}, \varphi) < P(\varphi)$ is a sharp condition for having a unique equilibrium state which is fully supported. If $P(\text{Sing}, \varphi) = P(\varphi)$, then φ has at least one equilibrium state supported on Sing .

For the class of potentials under consideration, Theorem A reduces the problem of uniqueness of equilibrium states to checking if the pressure gap $P(\text{Sing}, \varphi) < P(\varphi)$ holds. The following result establishes this gap, and hence uniqueness of equilibrium states, for a large class of Hölder continuous potentials.

Theorem B. *With \mathcal{F} and M be as above, let $\varphi: T^1M \rightarrow \mathbb{R}$ be a continuous function that is locally constant on a neighborhood of Sing . Then $P(\text{Sing}, \varphi) < P(\varphi)$.*

The case $\varphi = 0$ recovers Knieper's result that the singular set has smaller entropy than the whole system. In Knieper's work [16], this was obtained as a consequence of the uniqueness result. The argument presented here gives the first *direct* proof of the entropy gap.

We now state our results for the family of potentials $q\varphi^u$. In dimension 2, it is easy to check that $P(\text{Sing}, q\varphi^u) = 0$, and that $P(q\varphi^u) > 0$ for $q < 1$. Thus, the following result is a corollary of Theorem A.

Theorem C. *If M is a closed rank 1 surface, then the geodesic flow has a unique equilibrium state μ_q for the potential $q\varphi^u$ for each $q \in (-\infty, 1)$. This equilibrium state satisfies $\mu_q(\text{Reg}) = 1$, is fully supported, and is the weak* limit of weighted regular periodic orbits. Moreover, the function $q \mapsto P(q\varphi^u)$ is C^1 for $q \in (-\infty, 1)$.*

It follows from work of Ledrappier, Lima, and Sarig [18, 17] that these equilibrium states are Bernoulli, see §9. For rank 1 surfaces, this uniqueness result is optimal; any invariant measure supported on Sing is an equilibrium state for $q\varphi^u$ when $q \geq 1$. In higher dimensions, Sing can have positive entropy, but we can still exploit the entropy gap $h_{\text{top}}(\text{Sing}) < h_{\text{top}}(\mathcal{F})$. An easy argument, which we give in §9, gives the following result on $q\varphi^u$ for higher dimensional manifolds as a consequence of the entropy gap.

Theorem D. *Let \mathcal{F} be the geodesic flow for a closed rank 1 manifold. There exists $q_0 > 0$ such that the potential $q\varphi^u$ has a unique equilibrium state μ_q for each $q \in (-q_0, q_0)$. The function $q \mapsto P(q\varphi^u)$ is C^1 for $q \in (-q_0, q_0)$. Each μ_q gives full measure to Reg , is fully supported, and is obtained as the weak* limit of weighted regular periodic orbits.*

The entropy gap, and hence the q_0 provided by this theorem, may be arbitrarily small, see §9. If $h_{\text{top}}(\text{Sing}) = 0$, we observe in §9 that the gap holds on $(-q_0, 1)$, and in §10.2 we give a 3-dimensional M for which $\text{Sing} \neq \emptyset$ but the gap holds for all $q \in \mathbb{R}$. It is an open question whether the inequality $P(\text{Sing}, q\varphi^u) < P(q\varphi^u)$ always holds for all $q \in (-\infty, 1)$ when $\dim(M) > 2$.

As a final application, we prove in §10.1 that if the singular set is a finite union of periodic orbits, then our uniqueness results hold for C^0 -generic Hölder potentials; this includes the case when $\dim M = 2$ and the metric is real analytic.

The proof of Theorem A uses general machinery developed by the second and fourth authors [6], which was inspired by Bowen's work on uniqueness using the *expansivity* and *specification* properties [3] and its extension to flows by Franco [11]. The results in [6] use weaker versions of these properties which are formulated at the level of finite-length orbit segments; see §2.2. This allows us to avoid issues with asymptotic behavior of orbits that would be hard to control in our setting. The idea is that every orbit segment can be decomposed into 'good' and 'bad' parts, where the 'good' parts satisfy Bowen's conditions, and the 'bad' parts carry smaller topological pressure than the whole system.

Bowen's result applies to potentials satisfying a regularity condition that we call the *Bowen property*; our result uses the non-uniform Bowen property from [6], which holds here for all Hölder potentials. Verifying this condition for the potentials $q\varphi^u$ is a significant point in our argument. It is not currently known if horospheres are $C^{2+\alpha}$ for rank 1 manifolds in dimension greater than 2, which is necessary for Hölder continuity of the unstable distribution. Even in dimension 2, where horocycles are known to be $C^{2+\frac{1}{2}}$ by [14], Hölder continuity of the unstable distribution, and thus φ^u , is an open question.

The outline of the paper is as follows. In §2, we introduce background material, particularly the existence and uniqueness result from [6]. In §3, we state our most general theorem on equilibrium states for geodesic flow, Theorem 3.1. In §§4-6, we build up a proof of Theorem 3.1. In §7, we investigate regularity of the potentials $q\varphi^u$. In §8, we prove Theorem B. In §9, we complete the proofs of Theorems A, C, and D. In §10, we apply our results to some examples.

2. PRELIMINARIES

In this section we review definitions and results concerning pressure, specification, expansivity, geometry, and hyperbolicity.

2.1. Topological Pressure. Let X be a compact metric space, $\mathcal{F} = \{f_t\}$ a continuous flow on X , and $\varphi: X \rightarrow \mathbb{R}$ a continuous function. We denote the space of \mathcal{F} -invariant probability measures on X by $\mathcal{M}(\mathcal{F})$, and note that $\mathcal{M}(\mathcal{F}) = \bigcap_{t \in \mathbb{R}} \mathcal{M}(f_t)$. We denote the space of ergodic \mathcal{F} -invariant probability measures on X by $\mathcal{M}^e(\mathcal{F})$.

We recall the definition of the topological pressure of φ with respect to \mathcal{F} , referring the reader to [4, 22] for more background. For $\epsilon > 0$ and $t > 0$ the *Bowen ball of radius ϵ and order t* is

$$B_t(x, \epsilon) = \{y \in M \mid d(f_s x, f_s y) < \epsilon \text{ for all } 0 \leq s \leq t\}.$$

Given $\epsilon > 0$ and $t \in [0, \infty)$, a set $E \subset X$ is (t, ϵ) -*separated* if for all distinct $x, y \in E$ we have $y \notin \overline{B_t(x, \epsilon)}$.

We write $\Phi(x, t) = \int_0^t \varphi(f_s x) ds$ for the integral of φ along an orbit segment of length t . Let

$$(2.1) \quad \Lambda(\varphi, \epsilon, t) = \sup \left\{ \sum_{x \in E} e^{\Phi(x, t)} \mid E \subset X \text{ is } (t, \epsilon)\text{-separated} \right\}.$$

Then the *topological pressure of φ with respect to \mathcal{F}* is

$$P(\mathcal{F}, \varphi) = \lim_{\epsilon \rightarrow 0} \limsup_{t \rightarrow \infty} \frac{1}{t} \log \Lambda(\varphi, \epsilon, t).$$

The dependence on \mathcal{F} will usually be suppressed in the notation.

The *variational principle for pressure* states that if X is a compact metric space and \mathcal{F} is a continuous flow on X , then

$$P(\mathcal{F}, \varphi) = \sup_{\mu \in \mathcal{M}(\mathcal{F})} \left\{ h_\mu(\mathcal{F}) + \int \varphi d\mu \right\}.$$

A measure achieving the supremum is an *equilibrium state* for φ . If the entropy map $\mu \mapsto h_\mu$ is upper semi-continuous then equilibrium states exist for each continuous potential function. This is the case in our setting since the flow is C^∞ .

2.2. Criteria for uniqueness of equilibrium states. We review the general result proved by the second and fourth authors in [6] concerning the existence of a unique equilibrium state.

Given a flow (X, \mathcal{F}) , we think of $X \times [0, \infty)$ as the space of finite-length orbit segments by identifying (x, t) with $\{f_s(x) : 0 \leq s < t\}$.

Given $\mathcal{C} \subset X \times [0, \infty)$ and $t \geq 0$ we let $\mathcal{C}_t = \{x \in X : (x, t) \in \mathcal{C}\}$. The partition function associated to \mathcal{C} is

$$\Lambda(\mathcal{C}, \varphi, \delta, t) = \sup \left\{ \sum_{x \in E} e^{\Phi(x, t)} : E \subset \mathcal{C}_t \text{ is } (t, \delta)\text{-separated} \right\}.$$

When $\mathcal{C} = X \times [0, \infty)$ this reduces to (2.1). The *pressure of φ on \mathcal{C}* is

$$P(\mathcal{C}, \varphi) = \lim_{\delta \rightarrow 0} \limsup_{t \rightarrow \infty} \frac{1}{t} \log \Lambda(\mathcal{C}, \varphi, \delta, t).$$

For $\mathcal{C} = \emptyset$ we then define $P(\emptyset, \varphi) = -\infty$.

We can ask for the Bowen property and the specification property, defined below, to hold only on \mathcal{C} rather than the whole space.

Definition 2.1. *A collection $\mathcal{C} \subset X \times [0, \infty)$ of orbit segments has specification at scale $\rho > 0$ if there is $\tau = \tau(\rho)$ such that for every $(x_1, t_1), \dots, (x_N, t_N) \in \mathcal{C}$ there exist a point $y \in X$ and a sequence of times $\tau_1, \dots, \tau_{N-1} \in [0, \tau]$ such that for $s_0 = \tau_0 = 0$ and $s_j = \sum_{i=1}^j t_i + \sum_{i=1}^{j-1} \tau_i$, we have*

$$f_{s_{j-1} + \tau_{j-1}}(y) \in B_{t_j}(x_j, \rho)$$

for every $j \in \{1, \dots, N\}$. A collection $\mathcal{C} \subset X \times [0, \infty)$ has specification if it has specification at all scales. If $\mathcal{C} = X \times [0, \infty)$ has specification, then we say the flow has specification.

The definition above extends the specification property for the flow originally studied by Bowen, see [11, 15]. Even in the case $\mathcal{C} = X \times [0, \infty)$, this definition is weaker than Bowen's, see [6, §2.3].

Definition 2.2. *We say that $\varphi: X \rightarrow \mathbb{R}$ has the Bowen property on $\mathcal{C} \subset X \times [0, \infty)$ if there are $\epsilon, K > 0$ such that for all $(x, t) \in \mathcal{C}$ and $y \in B_t(x, \epsilon)$, we have $\sup_{y \in B_t(x, \epsilon)} |\Phi(x, t) - \Phi(y, t)| \leq K$.*

If φ has the Bowen property on $\mathcal{C} = X \times [0, \infty)$, then our definition agrees with the original definition of Bowen.

Definition 2.3. *A decomposition for $X \times [0, \infty)$ consists of three collections $\mathcal{P}, \mathcal{G}, \mathcal{S} \subset X \times [0, \infty)$ for which there exist three functions $p, g, s: X \times [0, \infty) \rightarrow [0, \infty)$ such that for every $(x, t) \in X \times [0, \infty)$, the values $p = p(x, t)$, $g = g(x, t)$, and $s = s(x, t)$ satisfy $t = p + g + s$, and*

$$(x, p) \in \mathcal{P}, \quad (f_p(x), g) \in \mathcal{G}, \quad (f_{p+g}(x), s) \in \mathcal{S}.$$

The conditions we are interested in depend only on the collections $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ rather than the functions p, g, s . However, we work with a fixed choice of (p, g, s) for the proof of the abstract theorem to apply.

We will construct a decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ such that \mathcal{G} has specification, the function φ has the Bowen property on \mathcal{G} , and the pressure on $[\mathcal{P}] \cup [\mathcal{S}]$ is less than the pressure of the entire system, where

$$[\mathcal{P}] := \{(x, n) \in X \times \mathbb{N} : (f_{-s}x, n + s + t) \in \mathcal{P} \text{ for some } s, t \in [0, 1]\}$$

and similarly for $[\mathcal{S}]$. The reason that we control the pressure of $[\mathcal{P}] \cup [\mathcal{S}]$ rather than the collection $\mathcal{P} \cup \mathcal{S}$ is a consequence of a technical step in the proof of the abstract result in [6] that required a passage from continuous to discrete time.

For $x \in X$ and $\epsilon > 0$ we let the *bi-infinite Bowen ball* be

$$\Gamma_\epsilon(x) = \{y \in X : d(f_t x, f_t y) \leq \epsilon \text{ for all } t \in \mathbb{R}\}.$$

Definition 2.4. *The set of non-expansive points at scale ϵ is*

$$\text{NE}(\epsilon) := \{x \in X \mid \Gamma_\epsilon(x) \not\subset f_{[-s,s]}(x) \text{ for any } s > 0\},$$

where $f_{[a,b]}(x) = \{f_t x : a \leq t \leq b\}$.

Definition 2.5. *Given a potential φ , the pressure of obstructions to expansivity is $P_{\text{exp}}^\perp(\varphi) := \lim_{\epsilon \rightarrow 0} P_{\text{exp}}^\perp(\varphi, \epsilon)$, where*

$$P_{\text{exp}}^\perp(\varphi, \epsilon) = \sup_{\mu \in \mathcal{M}^e(\mathcal{F})} \left\{ h_\mu(f_1) + \int \varphi d\mu : \mu(\text{NE}(\epsilon)) = 1 \right\}.$$

The point of this definition is that every ergodic measure whose free energy exceeds $P_{\text{exp}}^\perp(\varphi)$ gives zero measure to the non-expansive set, and thus “sees” only expansive behavior.

We can now state the abstract theorem that we will use to prove our uniqueness results.

Theorem 2.6. [6, Theorem A] *Let (X, \mathcal{F}) be a flow on a compact metric space, and $\varphi : X \rightarrow \mathbb{R}$ be a continuous potential function. Suppose that $P_{\text{exp}}^\perp(\varphi) < P(\varphi)$ and $X \times [0, \infty)$ admits a decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ with the following properties:*

- (I) \mathcal{G} has specification;
- (II) φ has the Bowen property on \mathcal{G} ;
- (III) $P([\mathcal{P}] \cup [\mathcal{S}], \varphi) < P(\varphi)$.

Then $(X, \mathcal{F}, \varphi)$ has a unique equilibrium state μ_φ .

2.3. Pressure and periodic orbits for geodesic flows. We define the *pressure of regular periodic orbits* for geodesic flow on a rank 1 manifold. This quantity was studied by Gelfert and Schapira [13], who called it the *Gurevic pressure*. It captures the exponential growth rate of regular closed geodesics, suitably weighted by the potential

function. Let $\text{Per}(T, \text{Reg})$ denote the set of prime closed geodesics of length bounded above by T which are contained in Reg . We define

$$(2.2) \quad P_{\text{Reg}}^*(\varphi) = \limsup_{T \rightarrow \infty} \frac{1}{T} \log \sum_{\gamma \in \text{Per}(T, \text{Reg})} e^{\Phi(\gamma)}$$

where $\Phi(\gamma)$ is the value given by integrating Φ around the closed geodesic (i.e. $\Phi(\gamma) := \Phi(v, |\gamma|)$ where $v \in T^1M$ is tangent to γ and $|\gamma|$ is the length of γ). It is easy to verify that in (2.2) we can instead sum over the set of prime closed geodesics of length between T and $T + \delta$, for any fixed $\delta > 0$. The pigeonhole principle yields the same upper exponential growth rate as in (2.2).

For a closed geodesic γ , let μ_γ be the normalized Lebesgue measure around the orbit. We say the weighted regular periodic orbits *equidistribute* to a measure μ if in the weak* topology we have

$$(2.3) \quad \mu = \lim_{T \rightarrow \infty} \frac{1}{C(T)} \sum_{\gamma \in \text{Per}(T, \text{Reg})} e^{\Phi(\gamma)} \mu_\gamma,$$

where $C(T)$ is the normalizing constant $\sum_{\gamma \in \text{Per}(T, \text{Reg})} \mu_\gamma(T^1M)$. This phenomenon was first investigated for equilibrium states in a uniformly hyperbolic setting by Parry [21]. In [13], Gelfert and Schapira observe that the proof of the variational principle shows that if $P_{\text{Reg}}^*(\varphi) = P(\varphi)$, then any weak* limit of $\frac{1}{C(T)} \sum_{\gamma \in \text{Per}(T, \text{Reg})} e^{\Phi(\gamma)} \mu_\gamma$ is an equilibrium state for φ . Thus if we know that $P_{\text{Reg}}^*(\varphi) = P(\varphi)$, and that φ has a unique equilibrium state μ , it follows immediately that the weighted regular periodic orbits equidistribute to μ .

In fact, Gelfert and Schapira explored a variety of definitions of topological pressure for geodesic flow on rank 1 manifolds [13], giving inequalities between four a priori different quantities, and giving a sufficient criterion for all of these to be equivalent. We refer the reader to [13] for details of all these quantities. The pressure of regular periodic orbits is the smallest of the four quantities they consider. Thus, if $P_{\text{Reg}}^*(\varphi) = P(\varphi)$, then all of the definitions of topological pressure considered by Gelfert and Schapira are equal.

2.4. Geometry. Throughout the paper M denotes a compact, connected, boundaryless smooth manifold with a smooth Riemannian metric g , all of whose sectional curvatures are nonpositive at every point.

For each $v \in TM$ there is a unique geodesic denoted γ_v such that $\dot{\gamma}(0) = v$. The *geodesic flow* $\mathcal{F} = (f_t)_{t \in \mathbb{R}}$ acts on TM by $f_t(v) = (\dot{\gamma}_v)(t)$. The unit tangent bundle T^1M is compact and \mathcal{F} -invariant; from now on we restrict to the flow on T^1M . We recall some well-known properties of geodesic flow in this setting; see [1, 9] for more details.

The Riemannian metric on M lifts to the *Sasaki metric* on TM . We write d_S for the distance function this Riemannian metric induces on T^1M . Another distance function on T^1M was used by Knieper in [16]:

$$(2.4) \quad d_K(v, w) = \max\{d(\gamma_v(t), \gamma_w(t)) \mid t \in [0, 1]\}.$$

We call d_K the *Knieper metric*; it need not be Riemannian. The two distance functions d_S and d_K are uniformly equivalent. We will typically consider Bowen balls with respect to the Knieper metric, so

$$\begin{aligned} B_T(v, \epsilon) &= \{w \in T^1M : d_K(f_tw, f_tv) < \epsilon \text{ for all } 0 \leq t \leq T\} \\ &= \{w \in T^1M : d(\gamma_w(t), \gamma_v(t)) < \epsilon \text{ for all } 0 \leq t \leq T + 1\}. \end{aligned}$$

A *Jacobi field* along a geodesic γ is a vector field along γ satisfying

$$(2.5) \quad J''(t) + R(J(t), \dot{\gamma}(t))\dot{\gamma}(t) = 0,$$

where R is the Riemannian curvature tensor on M and $'$ represents covariant differentiation along γ .

If $J(t)$ is a Jacobi field along a geodesic γ and both $J(t_0)$ and $J'(t_0)$ are orthogonal to $\dot{\gamma}(t_0)$ for some t_0 , then $J(t)$ and $J'(t)$ are orthogonal to $\dot{\gamma}(t)$ for all t . Such a Jacobi field is an *orthogonal Jacobi field*.

Nonpositivity of the sectional curvatures implies that $\|J(t)\|$ and $\|J(t)\|^2$ are convex functions of t ; this and related convexity properties will be useful in many places below.

For compact rank 1 manifolds, the set of vectors that have dense forward and backward orbits under \mathcal{F} is a dense G_δ set in T^1M . In particular, Reg is dense since it is open and invariant, and the geodesic flow is topologically transitive.

2.4.1. Invariant foliations. We describe three important \mathcal{F} -invariant subbundles E^u , E^s , and E^c of TT^1M . The bundle E^c is spanned by the vector field V that generates the flow \mathcal{F} . To describe E^u and E^s , we first write $\mathcal{J}(\gamma)$ for the space of orthogonal Jacobi fields for γ ; given $v \in T^1M$ there is a natural isomorphism $\xi \mapsto J_\xi$ between T_vT^1M and $\mathcal{J}(\gamma_v)$, which has the property that

$$(2.6) \quad \|df_t(\xi)\|^2 = \|J_\xi(t)\|^2 + \|J'_\xi(t)\|^2.$$

An orthogonal Jacobi field J along a geodesic γ is *stable* if $\|J(t)\|$ is bounded for $t \geq 0$, and *unstable* if it is bounded for $t \leq 0$. The stable and the unstable Jacobi fields each form linear subspaces of $\mathcal{J}(\gamma)$, which we denote by $\mathcal{J}^s(\gamma)$ and $\mathcal{J}^u(\gamma)$, respectively. The corresponding stable and unstable subbundles of TT^1M are

$$\begin{aligned} E^u(v) &= \{\xi \in T_v(T^1M) : J_\xi \in \mathcal{J}^u(\gamma_v)\}, \\ E^s(v) &= \{\xi \in T_v(T^1M) : J_\xi \in \mathcal{J}^s(\gamma_v)\}. \end{aligned}$$

The following properties are standard (see [9] for details):

- $\dim(E^u) = \dim(E^s) = n - 1$, and $\dim(E^c) = 1$;
- the subbundles are invariant under the geodesic flow;
- the subbundles depend continuously on v , see [9, 14];
- E^u and E^s are both orthogonal to E^c ;
- E^u and E^s intersect if and only if $v \in \text{Sing}$;
- E^σ is integrable to a foliation W^σ for each $\sigma \in \{u, s, cs, cu\}$;
- the foliations W^u and W^s are minimal [8, Theorem 6.1].

Almost every $v \in \text{Reg}$ (with respect to any invariant measure) has non-zero Lyapunov exponents and a corresponding Oseledec splitting $T_v T^1 M = E^s \oplus E^c \oplus E^u$ that agrees with the one above. However, in the general setting of Pesin theory, E^s and E^u are only measurable and do not extend beyond the regular set. Here they are continuous and globally defined, although since E^s and E^u are tangent on Sing , the subbundles do not define a splitting beyond the regular set.

In addition to the metrics d_S and d_K on $T^1 M$, we will need to consider for each $v \in T^1 M$ the *intrinsic metric* on $W^s(v)$ defined by

$$(2.7) \quad d^s(u, w) = \inf\{\ell(\pi\gamma) \mid \gamma: [0, 1] \rightarrow W^s(v), \gamma(0) = u, \gamma(1) = w\},$$

where $\pi: T^1 M \rightarrow M$ is the canonical projection, ℓ denotes length of the curve in M , and the infimum is over all C^1 curves γ connecting u and w in $W^s(v)$. In other words, $d^s(u, w)$ is the distance between the footprints $\pi(u)$ and $\pi(w)$ when we restrict ourselves to motion along the horosphere $H^s(v) = \pi W^s(v)$. Given $\rho > 0$, the *local stable leaf* through v of size ρ is

$$W_\rho^s(v) := \{w \in W^s(v) : d^s(v, w) \leq \rho\}.$$

Define d^u , $W_\rho^u(v)$ similarly. Locally, the intrinsic metric on $W^{cs}(v)$ is

$$d^{cs}(u, w) = |t| + d^s(f_t u, w),$$

where t is the unique value so $f_t u \in W^s(w)$. This extends to a metric on the whole leaf $W^{cs}(v)$. We define d^{cu} , $W_\rho^{cs}(v)$, $W_\rho^{cu}(v)$ in the obvious way. If we restrict ρ to be small, then the intrinsic metrics are uniformly equivalent to d_S and d_K . They have the useful property that given $v \in T^1 M$ and $\sigma \in \{s, cs\}$, the function $t \mapsto d^\sigma(f_t u, f_t w)$ is a convex nonincreasing function of t whenever $u, w \in W^\sigma(v)$. Similarly, this distance function is convex and nondecreasing when $\sigma \in \{u, cu\}$.

2.4.2. H -Jacobi fields and the function λ . Our hyperbolicity estimates will be given in terms of a function $\lambda: T^1 M \rightarrow [0, \infty)$, which we now describe. Let γ be a unit speed geodesic through $p \in M$, and let $H \subset M$ be a hypersurface orthogonal to γ at p . Let $\mathcal{J}_H(\gamma)$ be the set

of H -Jacobi fields obtained by varying γ through unit speed geodesics orthogonal to H . This is an $(n-1)$ -dimensional Lagrangian subspace of $\mathcal{J}(\gamma)$. Writing $H^{s,u}$ for the stable and unstable horospheres, we have $\mathcal{J}_{H^{s,u}}(\gamma) = \mathcal{J}^{s,u}(\gamma)$.

Let $\mathcal{U}: T_p H \rightarrow T_p H$ be the symmetric linear operator defined by $\mathcal{U}(v) = \nabla_v N$, where N is the field of unit vectors normal to H on the same side as $\dot{\gamma}(t_0)$; this determines the second fundamental form of H .

Lemma 2.7. *If J is an H -Jacobi field along γ , then $J'(t_0) = \mathcal{U}(J(t_0))$.*

Proof. Choose a variation $\alpha(s, t)$ of γ through unit speed geodesics such that $\alpha(s, t_0) \in H$ and $\frac{\nabla \alpha}{\partial s}(s, t_0)$ is a field of unit normals to H . Then

$$J'(t_0) = \frac{\nabla}{\partial t} \frac{\partial \alpha}{\partial s}(0, t_0) = \frac{\nabla}{\partial s} \frac{\partial \alpha}{\partial t}(0, t_0) = \nabla_{J(t_0)} N = \mathcal{U}(J(t_0)). \quad \square$$

The key consequence of Lemma 2.7 is that writing λ_H for the minimum eigenvalue of the linear map \mathcal{U} , every H -Jacobi field J has

$$(2.8) \quad \langle J, J' \rangle(t_0) = 2\langle J, \mathcal{U}J \rangle(t_0) \geq 2\lambda_H \langle J(t_0), J(t_0) \rangle,$$

which gives $(\log \|J\|^2)'(t_0) \geq 2\lambda_H$, and in particular

$$(2.9) \quad (\log \|J\|)'(t_0) \geq \lambda_H.$$

Let $\mathcal{U}^s(v): T_{\pi v} H^s \rightarrow T_{\pi v} H^s$ be the symmetric linear operator associated to the stable horosphere H^s , and similarly for \mathcal{U}^u . Then \mathcal{U}^u and \mathcal{U}^s are continuous, \mathcal{U}^u is positive semidefinite, \mathcal{U}^s is negative semidefinite, and $\mathcal{U}^u(-v) = -\mathcal{U}^s(v)$.

Let Λ be the maximum eigenvalue of $\mathcal{U}^u(v)$ for $v \in T^1 M$. If J_ξ is a stable or unstable Jacobi field we have $\|J'_\xi(t)\| \leq \Lambda \|J_\xi(t)\|$ for all t . Thus if ξ is in E^s or E^u , then by (2.6) and Lemma 2.7, $\|df_t \xi\|$ and $\|J_\xi(t)\|$ are uniformly comparable in the sense that

$$(2.10) \quad \|J_\xi(t)\|^2 \leq \|df_t \xi\|^2 \leq (1 + \Lambda^2) \|J_\xi(t)\|^2.$$

Definition 2.8. *For $v \in T^1 M$, let $\lambda^u(v)$ be the minimum eigenvalue of $\mathcal{U}^u(v)$ and let $\lambda^s(v) = \lambda^u(-v)$. Let $\lambda(v) = \min(\lambda^u(v), \lambda^s(v))$.*

The functions λ^u , λ^s , and λ are continuous by continuity of $\mathcal{U}^u(v)$. By positive (negative) semidefiniteness of $\mathcal{U}^{u,s}$, we have $\lambda^{u,s} \geq 0$. The following is an immediate consequence of (2.9).

Lemma 2.9. *Given $v \in T^1 M$, let J^u be an unstable Jacobi field along γ_v and J^s be a stable Jacobi field along γ_v . Then*

$$\|J^u(T)\| \geq e^{\int_0^T \lambda^u(f_t v) dt} \|J^u(0)\| \quad \text{and} \quad \|J^s(T)\| \leq e^{-\int_0^T \lambda^s(f_t v) dt} \|J^s(0)\|.$$

In §3.2 we collect some more properties of the functions λ , λ^s , λ^u .

3. DECOMPOSITIONS FOR GEODESIC FLOW

3.1. Main theorem. Now we state our main uniqueness result, which we apply to obtain Theorem A.

Theorem 3.1. *Let $\varphi: T^1M \rightarrow \mathbb{R}$ be continuous. If $P(\text{Sing}, \varphi) < P(\varphi)$, and for all $\eta > 0$ the potential φ has the Bowen property on*

$$\mathcal{G}(\eta) = \left\{ (v, t) : \int_0^\tau \lambda(f_s v) ds \geq \eta\tau, \int_0^\tau \lambda(f_{-s} f_t v) ds \geq \eta\tau \quad \forall \tau \in [0, t] \right\},$$

then the geodesic flow has a unique equilibrium state for φ . This equilibrium state is fully supported, has $\mu(\text{Reg}) = 1$, and is the weak limit of weighted regular periodic orbit measures as in (2.3).*

The set of potentials having the Bowen property on $\mathcal{G}(\eta)$ for all $\eta > 0$ contains all Hölder potentials, all scalar multiples of the geometric potential, and all linear combinations of such potentials; see §7.

We build up a proof of Theorem 3.1 in the next few sections. We start by describing the decomposition we use to apply Theorem 2.6.

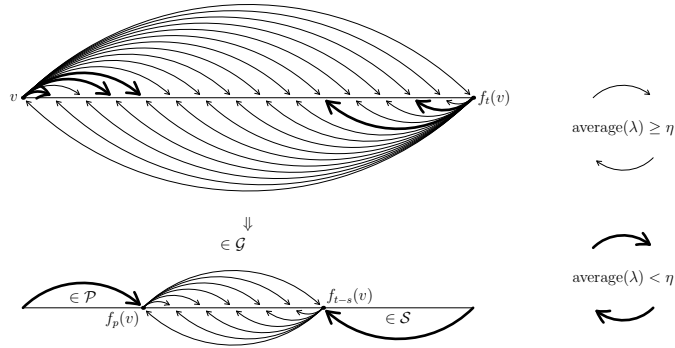


FIGURE 3.1. Decomposing an orbit segment.

Given $\eta > 0$, let $\mathcal{B}(\eta) := \{(v, T) : \int_0^T \lambda(f_t v) dt < \eta T\}$. We define maps $p, g, s, : X \times [0, \infty) \rightarrow [0, \infty)$. Given an orbit segment (v, t) , take $p = p(v, t)$ to be the largest time such that $(v, p) \in \mathcal{B}(\eta)$. Let $s = s(v, t)$ be the largest time in $[0, t - p]$ such that the orbit segment $(f_{t-s}(v), s)$ is in $\mathcal{B}(\eta)$. The function g determines the remaining part of the orbit segment denoted $(f_p v, g)$, so $g = t - p - s$. It is easily checked that $(f_p v, g) \in \mathcal{G}(\eta)$. Thus the triple $(\mathcal{B}(\eta), \mathcal{G}(\eta), \mathcal{B}(\eta))$ equipped with the functions (p, g, s) determines a decomposition for $X \times [0, \infty)$ in the sense of Definition 2.3. We will show that if $P(\text{Sing}, \varphi) < P(\varphi)$ and if $\eta > 0$ is chosen sufficiently small, then the hypotheses of Theorem 2.6 are satisfied using the decomposition $(\mathcal{B}(\eta), \mathcal{G}(\eta), \mathcal{B}(\eta))$. This will guarantee uniqueness of the equilibrium state.

3.2. Properties of λ .

Lemma 3.2. *Given a geodesic γ , the following are equivalent:*

- $\lambda^u(\gamma'(0)) = 0$;
- *there is a nontrivial orthogonal Jacobi field J on γ such that $J(t)$ is constant for all $t \leq 0$.*

A similar statement holds for λ^s and $t \geq 0$.

Proof. The backward direction is immediate, so we prove the forward. If $\lambda^u(v) = 0$, then there is a nonzero $w \in T_{\gamma(0)}H^u(v)$ with $\mathcal{U}^u(w) = 0$. The corresponding $H^u(v)$ -Jacobi field has $J'(0) = 0$ (Lemma 2.7) and is bounded for $t \leq 0$, so by convexity $\|J(t)\|$ is constant for $t \leq 0$. For such t this gives $0 = \langle J', J \rangle = \langle \mathcal{U}_{f_tv}^u J, J \rangle$, hence $\mathcal{U}_{f_tv}^u J = 0$ since \mathcal{U}^u is positive semidefinite symmetric, so $J(t)$ is constant for $t \leq 0$. \square

In particular, Lemma 3.2 shows that if $\lambda^s(v) = 0$ then $\lambda^s(f_tv) = 0$ for all $t \geq 0$, and similarly for λ^u with $t \leq 0$.

Lemma 3.3. *The following are equivalent for $v \in T^1M$.*

- (a) $v \in \text{Sing}$.
- (b) $\lambda^s(f_tv) = 0$ for all $t \in \mathbb{R}$.
- (c) $\lambda^u(f_tv) = 0$ for all $t \in \mathbb{R}$.

Proof. If $v \in \text{Sing}$, then there is a parallel Jacobi field $J(t)$ along $\gamma_v(t)$, which gives $\lambda^s = \lambda^u = 0$. Since Sing is invariant, this gives (b) and (c).

Now we show that (b) implies (a). If $\lambda^s(f_tv) = 0$ for every $t \in \mathbb{R}$, then for every $T \geq 0$ there is a stable Jacobi field J_T along γ_v that is constant (with unit length) for $t \geq -T$. By compactness we get a sequence $T_k \rightarrow \infty$ for which $J_{T_k}(0)$ and $J'_{T_k}(0)$ converge to some $J(0), J'(0) \in T_{\pi v}M$; the corresponding Jacobi field J is constant for all time, so $v \in \text{Sing}$. The proof that (c) implies (a) is similar. \square

The following is an immediate consequence of Lemmas 3.2 and 3.3.

Corollary 3.4. *The function $\lambda: T^1M \rightarrow [0, \infty)$ vanishes on Sing . If $\lambda(v) = 0$, then there is a nontrivial orthogonal Jacobi field J on γ such that $J(t)$ is constant for all $t \leq 0$ or for all $t \geq 0$.*

We also have the following quantitative version of Lemma 3.3, and two corollaries which are useful for our topological pressure estimates.

Proposition 3.5. *For any $\delta > 0$, there are $\eta > 0$ and $T > 0$ such that if $\lambda^s(f_tv) \leq \eta$ for all $t \in [-T, T]$, then $d_K(v, \text{Sing}) < \delta$. A similar result holds for λ^u .*

Proof. Given $T, \eta > 0$, consider the open set $A(T, \eta) = \{v \in T^1M : \lambda^s(f_tv) > \eta \text{ for some } t \in [-T, T]\}$. Let $K = \{v : d_K(v, \text{Sing}) \geq \delta\}$. By compactness and Lemma 3.3 there are T, η such that $K \subset A(T, \eta)$. \square

Corollary 3.6. *Let $\lambda(v) = 0$. Then $d_K(f_tv, \text{Sing}) \rightarrow 0$ as $t \rightarrow \infty$ or $d_K(f_tv, \text{Sing}) \rightarrow 0$ as $t \rightarrow -\infty$.*

Proof. Suppose $\lambda^s(v) = 0$; the case $\lambda^u(v) = 0$ is similar. Given $\delta > 0$, by Proposition 3.5 there are $\eta, T > 0$ such that if $\lambda^s(f_tw) \leq \eta$ for all $t \in [-T, T]$, then $d_K(w, \text{Sing}) < \delta$. Thus for every $\tau \geq T$, we can put $w = f_\tau v$ and conclude that $d_K(f_\tau v, \text{Sing}) < \delta$. \square

Corollary 3.7. *Let μ be an invariant measure such that $\lambda(v) = 0$ for μ almost every v . Then $\text{supp}(\mu) \subset \text{Sing}$.*

Proof. By Corollary 3.6, if $\lambda(v) = 0$ and $v \in \text{Reg}$, then v cannot be both forward recurrent and backward recurrent. Since μ -a.e. v is both forward and backward recurrent, we see that $\mu(\text{Reg}) = 0$. \square

3.3. Uniform estimates on $\mathcal{G}(\eta)$. To go from the Jacobi field estimates in Lemma 2.9 to local estimates near orbit segments in $\mathcal{G}(\eta)$, we can use uniform continuity of λ : given $\eta > 0$, let $\delta = \delta(\eta) > 0$ be small enough that if $v, w \in T^1M$ have $d_K(v, w) < \delta e^\Lambda$, then $|\lambda(v) - \lambda(w)| \leq \frac{\eta}{2}$. In particular, this applies if $w \in W_\delta^s(v)$ or $w \in W_\delta^u(v)$. Define $\tilde{\lambda} : T^1M \rightarrow [0, \infty)$ by $\tilde{\lambda}(v) = \max(0, \lambda(v) - \frac{\eta}{2})$, and observe that

$$(3.1) \quad \lambda(w) \geq \tilde{\lambda}(v) \text{ for every } v, w \in T^1M \text{ with } d_K(v, w) < \delta.$$

In particular, if $w \in B_T(v, \delta)$ then

$$(3.2) \quad \int_0^T \lambda(f_tw) dt \geq \int_0^T \tilde{\lambda}(f_tv) dt \geq \int_0^T \lambda(f_tv) dt - \frac{\eta}{2}T.$$

Now we can integrate the Jacobi field estimates.

Lemma 3.8. *Given η, δ as in (3.1), $v \in T^1M$, and $w, w' \in W_\delta^s(v)$, we have the following for every $t \geq 0$:*

$$(3.3) \quad d^s(f_tw, f_tw') \leq d^s(w, w') e^{-\int_0^t \tilde{\lambda}(f_\tau v) d\tau}.$$

Similarly, if $w, w' \in W_\delta^u(v)$, then for every $t \geq 0$ we have

$$(3.4) \quad d^u(f_{-t}w, f_{-t}w') \leq d^u(w, w') e^{-\int_0^t \tilde{\lambda}(f_{-\tau}v) d\tau}.$$

Proof. We prove (3.3); (3.4) is similar. Recalling the definition of d^s in (2.7), let $\gamma : [0, 1] \rightarrow W_\delta^s(v)$ be a curve that connects w and w' ; then $f_t\gamma$ is a curve on $W^s(f_tv)$ connecting f_tw and f_tw' , and we want to compare the lengths $\ell(\pi\gamma)$ and $\ell(\pi f_t\gamma)$. For each $r \in [0, 1]$, the vector $\gamma(r) \in T^1M$ determines a geodesic ζ_r that is normal to the

stable horosphere $\pi W_\delta^s(v)$; this one-parameter family of geodesics gives a family of stable Jacobi fields $J_r \in \mathcal{J}^s(\zeta_r)$. By Lemma 2.9, these satisfy

$$\|J_r(t)\| \leq e^{-\int_0^t \lambda(\dot{\zeta}_r(\tau)) d\tau} \|J_r(0)\| \leq e^{-\int_0^t \tilde{\lambda}(f_\tau v) d\tau} \|J_r(0)\|,$$

and integrating over $r \in [0, 1]$ gives $\ell(\pi f_t \gamma) \leq e^{-\int_0^t \tilde{\lambda}(f_\tau v) d\tau} \ell(\pi \gamma)$. By (2.7), taking an infimum over all such γ gives (3.3). \square

When $(v, T) \in \mathcal{G}(\eta)$, the following lemma is an immediate consequence of (3.1), (3.2), and Lemma 3.8.

Lemma 3.9. *Given η, δ as in (3.1) and $(v, T) \in \mathcal{G}(\eta)$, every $w \in B_T(v, \delta)$ has $(w, T) \in \mathcal{G}(\frac{\eta}{2})$. Moreover, for every $w, w' \in W_\delta^s(v)$ and $0 \leq t \leq T$ we have*

$$(3.5) \quad d^s(f_t w, f_t w') \leq d^s(w, w') e^{-\frac{\eta}{2} t},$$

and for every $w, w' \in f_{-T} W_\delta^u(f_T v)$ and $0 \leq t \leq T$, we have

$$(3.6) \quad d^u(f_t w, f_t w') \leq d^u(f_T w, f_T w') e^{-\frac{\eta}{2} (T-t)}.$$

3.4. Uniformly regular points. For $\eta > 0$, we define

$$(3.7) \quad \text{Reg}(\eta) = \{v : \lambda(v) \geq \eta\}.$$

Note that if $(v, t) \in \mathcal{G}(\eta)$ for some $t > 0$, then $\lambda(v) \geq \eta$ and $\lambda(f_t v) \geq \eta$, and thus $v \in \text{Reg}(\eta)$ and $f_t v \in \text{Reg}(\eta)$. Note that $\text{Reg}(\eta_1) \subset \text{Reg}(\eta_2)$ if $\eta_1 \geq \eta_2$ and each $\text{Reg}(\eta)$ is compact.

Lemma 3.10. *For all $\eta > 0$, there exists $\theta > 0$ so that for any $v \in \text{Reg}(\eta)$, we have $\angle(E^u(v), E^s(v)) \geq \theta$.*

Proof. The angle is continuous in v and positive on $\text{Reg}(\eta)$. \square

Lemma 3.11. $\{v : \lambda(v) > 0\} = \bigcup_{\eta > 0} \text{Reg}(\eta)$ is dense in $T^1 M$.

Proof. Let v be a point whose forward and backward orbits are both dense. By Corollary 3.6 we have $\lambda(v) > 0$, and the same is true for every $f_t v$. Since $\{f_t v : t \in \mathbb{R}\}$ is dense in $T^1 M$, we are done. \square

Lemma 3.12. *Suppose $v \in \text{Reg}(\eta)$. Let J^u be an unstable Jacobi field along γ_v , and J^s be a stable Jacobi field along γ_v . For all $t \geq 0$,*

$$\|J^u(t)\| \geq (1 + \eta t) \|J^u(0)\| \quad \text{and} \quad \|J^s(-t)\| \geq (1 + \eta t) \|J^s(0)\|.$$

Proof. Since $v \in \text{Reg}(\eta)$, (2.9) gives $\|J^u\|'(0) \geq \eta \|J^u(0)\|$. Convexity of $\|J^u(t)\|$ gives the first inequality; the second is similar. \square

The following proposition and corollary play a crucial role of our proof of the pressure gap in §8.

Proposition 3.13. *For any $R, \epsilon, \eta > 0$ there exists $T > 0$ such that if $w \in \text{Reg}(\eta)$ and $f_T(v) \in W_R^u(f_T w)$, then $v \in W_\epsilon^u(w)$. Similarly, if $w \in \text{Reg}(\eta)$ and $f_{-T}(v) \in W_R^s(f_{-T} w)$, then $v \in W_\epsilon^s(w)$.*

Proof. We prove the first assertion; the second is similar. Fix $\eta > 0$ and let $\delta > 0$ be as in (3.1). Given $w \in \text{Reg}(\eta)$ and $v \in W^u(w)$, let $\rho(t) = d^u(f_t v, f_t w)$. We claim that when $T > \frac{2R}{\eta\delta}$, we have $\rho(0) \leq \frac{2R}{T\eta}$; this will prove the proposition by taking $T > \max(\frac{2R}{\eta\delta}, \frac{2R}{\eta\epsilon})$.

To prove the claim, let $\gamma: [0, 1] \rightarrow W^u(f_T w)$ be a curve connecting $f_T w$ to $f_T v$; let $\rho_\gamma(t) = \ell(f_{t-T}\gamma)$, so that $\rho(t) = \inf_\gamma \rho_\gamma(t)$. The geodesics ζ_r tangent to $\gamma(r)$ determine a family of unstable Jacobi fields $J_r \in \mathcal{J}^u(\zeta_r)$ such that $\rho_\gamma(t) = \int_0^1 \|J_r(t)\| dr$, and hence

$$(3.8) \quad \rho'_\gamma(t) = \int_0^1 \|J_r\|'(t) dr \geq \int_0^1 \lambda(\dot{\zeta}_r(t)) \|J_r(t)\| dr$$

When $t = 0$ there are two possibilities: either $f_{-T}\gamma$ is contained in $W_\delta^u(w)$, or it leaves it at some point. In the first case, $\lambda(\dot{\zeta}_r(0)) > \frac{\eta}{2}$ for every $r \in [0, 1]$ by (3.1), so (3.8) gives $\rho'_\gamma(0) \geq \frac{\eta}{2}\rho_\gamma(0)$. In the second case, let $r_0 = \sup\{r_1 : f_{-T}(\gamma(r)) \in W_\delta^u(w) \text{ for every } r \in [0, r_1]\}$; then (3.1) and (3.8) give $\rho'_\gamma(0) \geq \int_0^{r_0} \frac{\eta}{2} \|J_r(0)\| dr \geq \frac{\eta}{2}\delta$. Convexity gives

$$\rho'_\gamma(t) \geq \rho'_\gamma(0) \geq \frac{\eta}{2} \min(\delta, \rho_\gamma(0))$$

for every $t \geq 0$. In particular, we have

$$\rho_\gamma(T) \geq T\rho'_\gamma(0) \geq \frac{\eta}{2}T \min(\delta, \rho_\gamma(0)),$$

and taking an infimum over all γ we see that either $R \geq \frac{\eta}{2}T\delta$ or $R \geq \frac{\eta}{2}T\rho(0)$, which proves the claim from the first paragraph. \square

Corollary 3.14. *For every $R > 0$ and $\eta > \eta' > 0$, there is $T > 0$ such that given any $v, w \in T^1M$ with either $f_T(v) \in W_R^u(f_T w)$ or $f_{-T}(v) \in W_R^s(f_{-T} w)$, we have $\lambda^u(v) \geq \eta \Rightarrow \lambda^u(w) \geq \eta'$. Equivalently, we have $\lambda^u(w) < \eta' \Rightarrow \lambda^u(v) < \eta$.*

Proof. By uniform continuity of λ^u , we can take ϵ sufficiently small that if $v \in W_\epsilon^\sigma(w)$ for $\sigma \in \{s, u\}$, and $\lambda^u(w) \geq \eta$, then $\lambda^u(v) \geq \eta'$. Suppose that $f_T(v) \in W_R^u(f_T w)$ and $w \in \text{Reg}(\eta)$. Then $\lambda^u(w) \geq \lambda(w) \geq \eta$. By Proposition 3.13, $v \in W_\epsilon^u(w)$, and thus $\lambda^u(v) \geq \eta'$. Thus, if $\lambda^u(w) \geq \eta$, then $\lambda^u(v) \geq \eta'$. The argument when $f_{-T}(v) \in W_R^s(f_{-T} w)$ is analogous. \square

A similar argument shows that under the hypotheses of the corollary, $\lambda^s(v) < \eta' \Rightarrow \lambda^s(w) < \eta$, and $\lambda(v) < \eta' \Rightarrow \lambda(w) < \eta$, although we will not need these statements in our analysis.

4. THE SPECIFICATION PROPERTY

This section builds up a proof of the following result, which verifies condition (I) from Theorem 2.6 by proving specification for $\mathcal{G}(\eta)$.

Theorem 4.1. *For geodesic flow on a rank 1 manifold, let $\mathcal{C}(\eta)$ be the set of orbit segments that both start and end in $\text{Reg}(\eta)$. Then $\mathcal{C}(\eta)$ has the specification property. In particular, since $\mathcal{G}(\eta) \subset \mathcal{C}(\eta)$, it follows that $\mathcal{G}(\eta)$ has the specification property.*

The proof is based on uniformity of the local product structure for the foliations W^u , W^{cs} at the endpoints of orbits in $\mathcal{C}(\eta)$. To make this idea precise, we define local product structure at a point for a fixed scale and distortion constant. We work with the Knieper metric d_K from (2.4) and the leafwise metrics d^s and d^u from (2.7). In what follows, $B(v, \delta)$ denotes the ball in the Knieper metric d_K .

Definition 4.2. *The foliations W^u , W^{cs} have local product structure (LPS) at scale $\delta > 0$ with constant $\kappa \geq 1$ at $v \in T^1M$ if for all $w_1, w_2 \in B(v, \delta)$, the intersection $W_{\kappa\delta}^u(w_1) \cap W_{\kappa\delta}^{cs}(w_2)$ contains a single point, which we denote by $[w_1, w_2]$, and if moreover we have*

$$\begin{aligned} d^u(w_1, [w_1, w_2]) &\leq \kappa d_K(w_1, w_2), \\ d^{cs}(w_2, [w_1, w_2]) &\leq \kappa d_K(w_1, w_2). \end{aligned}$$

If W^u , W^{cs} have LPS at scale δ with constant κ at $v \in T^1M$, then for every $\epsilon \in (0, \delta]$, they have LPS at scale ϵ with constant κ at v . Also, they have LPS at scale $\delta/2$ with constant κ at every $w \in B(v, \delta/2)$.

We control d_K in terms of d^u and d^{cs} . Given $v \in T^1M$ and $w \in W^{cs}(v)$, the function $t \mapsto d^{cs}(f_tv, f_tw)$ is non-increasing, so (2.4) gives

$$(4.1) \quad d_K(v, w) \leq d^{cs}(v, w).$$

Moreover, writing $d_t(v, w) = \sup_{\tau \in [0, t]} d_K(v, w)$, monotonicity gives

$$(4.2) \quad d_t(v, w) \leq d^{cs}(v, w).$$

For $w \in W^u(v)$, we use (2.9) and the argument of Lemma 3.9 to get

$$(4.3) \quad \begin{aligned} d_K(v, w) &\leq e^\Lambda d^u(v, w), \\ d_t(v, w) &\leq d^u(f_{t+1}v, f_{t+1}w) \leq e^\Lambda d^u(f_tv, f_tw) \end{aligned}$$

where Λ is as defined in §2.4.2.

We need a lemma on uniform density of W^u at points where the foliations have LPS at a fixed scale and distortion constant.

Lemma 4.3. *Given $\kappa \geq 1$ and $\epsilon > 0$, there exists $T = T(\kappa, \epsilon)$ so that if W^u, W^{cs} have LPS at scale ϵ with constant κ at $v, w \in T^1M$, then*

$$\left(\bigcup_{0 \leq t \leq T} f_t(W_\epsilon^u(v)) \right) \cap W_\epsilon^{cs}(w) \neq \emptyset.$$

The proof uses the following consequence of transitivity, which is proved by a standard compactness argument.

Claim 4.4. *Let \mathcal{F} be a continuous flow on a compact metric space X , and let $x_0 \in X$ have dense forward orbit. Then for all $\epsilon > 0$, there exists T such that for all $x, y \in X$, there exists a point \bar{x} on the forward orbit of x_0 and a time $t \in [0, T]$ so that $\bar{x} \in B(x, \epsilon)$ and $f_t \bar{x} \in B(y, \epsilon)$.*

Proof of Lemma 4.3. Let $\epsilon' = \epsilon/(4\kappa^2)$, and use Claim 4.4 to find $T = T(\epsilon')$ and (\bar{v}, τ) with $\tau \in [0, T]$ so that $\bar{v} \in B(v, \epsilon')$ and $f_\tau \bar{v} \in B(w, \epsilon')$.

Let $u_1 = [v, \bar{v}]$, i.e. $u_1 \in W_{\kappa\epsilon'}^u(v) \cap W_{\kappa\epsilon'}^{cs}(\bar{v})$. Since $u_1 \in W_{\kappa\epsilon'}^{cs}(\bar{v})$,

$$f_\tau u_1 \in B(f_\tau \bar{v}, \kappa\epsilon') \subset B(w, 2\kappa\epsilon').$$

Let $u_2 = [f_\tau u_1, w]$, i.e. $u_2 \in W_{\epsilon/2}^u(f_\tau u_1) \cap W_{\epsilon/2}^{cs}(w)$, where we recall that $\epsilon/2 = 2\kappa^2\epsilon'$. Then we have $f_{-\tau} u_2 \in W_{\epsilon/2}^u(u_1)$, and since $u_1 \in W_{\kappa\epsilon'}^u(v) \subset W_{\epsilon/2}^u(v)$, it follows that $f_{-\tau} u_2 \in W_\epsilon^u(v)$. Thus u_2 is in the intersection we want to show is non-empty. \square

Corollary 4.5. *Given $\eta > 0$, there exists $\delta > 0$ so that if $v, w \in \text{Reg}(\eta)$, and v', w' satisfy $d_K(v, v') < \delta$, and $d_K(w, w') < \delta$, then for any $\rho \in (0, \delta]$, there exists T so that*

$$\left(\bigcup_{0 \leq t \leq T} f_t(W_\rho^u(v')) \right) \cap W_\rho^{cs}(w') \neq \emptyset.$$

Proof. Lemma 3.10 gives a uniform lower bound on the angle of intersection of W^u and W^{cs} for $v \in \text{Reg}(\eta)$, so there are $\delta > 0$ and $\kappa \geq 1$ such that at every $v \in \text{Reg}(\eta)$, W^u and W^{cs} have LPS at scale 2δ with constant κ . Thus, at v', w' , W^u and W^{cs} have LPS at scale ρ with constant κ for any $\rho \in (0, \delta]$. Thus, Lemma 4.3 applies. \square

In particular, if $(v, s), (w, t) \in \mathcal{C}(\eta)$, then Corollary 4.5 applies at the points $f_s v, w$. We are now ready to prove the specification property on $\mathcal{C}(\eta)$. First fix $(v_0, t_0) \in \mathcal{C}(\eta)$ with $t_0 \geq 1$, and let $\epsilon > 0$ be small enough that $\lambda \geq \eta/2$ on $W_\epsilon^s(v_0)$ and $W_\epsilon^u(f_{t_0} v_0)$. Let $\alpha = 1 + \eta t_0/2$; it follows from Lemma 3.12 and the arguments in §3.3 that for every $w, w' \in f_{-t_0} W_\epsilon^u(f_{t_0} v_0)$ we have

$$(4.4) \quad d^u(f_{t_0} w, f_{t_0} w') \geq \alpha d^u(w, w').$$

Fix $0 < \rho < \min(\delta, \epsilon)$, and let $\rho' = \rho / (6e^\Lambda \sum_{i=1}^\infty \alpha^{-i})$. Take T given by Corollary 4.5 so that $(\bigcup_{0 \leq t \leq T} f_t(W_{\rho'}^u(v))) \cap W_{\rho'}^{cs}(w) \neq \emptyset$ whenever v, w are within distance δ of points in $\text{Reg}(\eta)$. We show that $\mathcal{C}(\eta)$ has specification at scale ρ with transition time $2T + t_0$.

Given any $(v_1, t_1), \dots, (v_k, t_k) \in \mathcal{C}(\eta)$, we construct orbit segments (w_j, s_j) iteratively such that the orbit segment (w_j, s_j) shadows first (v_1, t_1) , then (v_0, t_0) , then (v_2, t_2) , then (v_0, t_0) , then (v_3, t_3) , and so on up through (v_j, t_j) .

Start by letting $w_1 = v_1$ and $s_1 = t_1$. Applying Corollary 4.5 at $f_{s_1}w_1$ and v_0 gives $\tau_1 \in [0, T]$ such that $(f_{\tau_1}(W_{\rho'}^u(f_{s_1}w_1))) \cap W_{\rho'}^{cs}(v_0) \neq \emptyset$; in particular, there is u_1 such that

$$f_{s_1}u_1 \in W_{\rho'}^u(f_{s_1}w_1) \quad \text{and} \quad f_{s_1+\tau_1}u_1 \in W_{\rho'}^{cs}(v_0).$$

Now applying Corollary 4.5 at $f_{s_1+\tau_1+t_0}u_1$ and v_2 , we get $\tau'_1 \in [0, T]$ and w_2 such that

$$f_{s_1+\tau_1+t_0}w_2 \in W_{\rho'}^u(f_{s_1+\tau_1+t_0}u_1) \quad \text{and} \quad f_{s_1+\tau_1+t_0+\tau'_1}w_2 \in W_{\rho'}^{cs}(v_2).$$

We continue this procedure recursively to obtain a sequence of points w_j, u_j . That is, we produce points w_j, u_j and times τ_j, τ'_j such that writing $s'_j = s_j + \tau_j + t_0$ and $s_{j+1} = s'_j + \tau'_j + t_{j+1}$, we have

$$(4.5) \quad f_{s_j}(u_j) \in W_{\rho'}^u(f_{s_j}(w_j)) \quad \text{and} \quad f_{s_j+\tau_j}(u_j) \in W_{\rho'}^{cs}(v_0),$$

$$(4.6) \quad f_{s'_j}(w_{j+1}) \in W_{\rho'}^u(f_{s'_j}(u_j)) \quad \text{and} \quad f_{s'_j+\tau'_j}(w_{j+1}) \in W_{\rho'}^{cs}(v_{j+1}).$$

To guarantee that such points and times exist for all $1 \leq j \leq k$, we observe that once w_j is chosen, we have $f_{s_j}w_j \in W_{\rho'}^{cs}(f_{t_j}v_j)$. Since $\rho' < \delta$, then $d_K(f_{s_j}w_j, f_{t_j}v_j) < \delta$. Thus, Corollary 4.5 applies to give the existence of u_j and τ_j satisfying (4.5). Once u_j is chosen, the same argument shows that there are w_{j+1} and τ'_j satisfying (4.5).

We show that (w_k, s_k) is the orbit we want for the specification property. All of the points w_j, u_j lie on $W^u(v_1)$. We have expansion by a factor of α whenever the orbit passes near (v_0, t_0) ; thus by (4.5) and the fact that d^u is non-increasing in backwards time, we have

$$\begin{aligned} \rho' &\geq d^u(f_{s_j}u_j, f_{s_j}w_j) \geq \alpha d^u(f_{s_{j-1}}u_j, f_{s_{j-1}}w_j) \\ &\geq \dots \geq \alpha^j d^u(f_{s_1}u_j, f_{s_1}w_j). \end{aligned}$$

Similarly, (4.6) and α -expansion gives $d^u(f_{s_j}w_{j+1}, f_{s_j}u_j) \leq \rho'\alpha^{-1}$. Iterating, we obtain the following estimates for all $1 \leq i \leq j$:

$$(4.7) \quad \begin{aligned} d^u(f_{s_i}u_j, f_{s_i}w_j) &\leq \rho'\alpha^{-(j-i)}, \\ d^u(f_{s_i}w_{j+1}, f_{s_i}u_j) &\leq \rho'\alpha^{-(1+j-i)}. \end{aligned}$$

Summing these gives

$$\begin{aligned} d^u(f_{s_i} w_j, f_{s_i} w_i) &\leq \sum_{\ell=i}^{j-1} d^u(f_{s_i} w_{\ell+1}, f_{s_i} u_\ell) + d^u(f_{s_i} u_\ell, f_{s_i} w_\ell) \\ &\leq \sum_{\ell=i}^{j-1} \rho' (\alpha^{-(\ell-i)} + \alpha^{-(1+\ell-i)}) \leq 2\rho' \sum_{n=0}^{\infty} \alpha^{-n} = \frac{\rho}{3e^\Lambda}. \end{aligned}$$

Together with (4.3), this gives

$$d_{t_i}(f_{s_i-t_i} w_j, f_{s_i-t_i} w_i) \leq e^\Lambda d^u(f_{s_i} w_j, f_{s_i} w_i) \leq \rho/3.$$

Recall from (4.6) that $f_{s_i-t_i} w_i = f_{s'_{i-1}+\tau'_{i-1}} w_i \in W_{\rho'}^{cs}(v_i)$, so (4.2) gives $d_{t_i}(f_{s_i-t_i} w_i, v_i) \leq \rho'$. Summing the two bounds gives $d_{t_i}(f_{s_i-t_i} w_j, v_i) < \rho$. In the case $j = k$, this shows that (w_k, s_k) is the orbit required for specification at scale ρ ; the transition times are $\tau_j + t_0 + \tau'_j \in [0, 2T + t_0]$. Since ρ can be taken arbitrarily small, this proves Theorem 4.1.

With a little modification, the proof of Theorem 4.1 yields the following result, which we will need in §8.

Proposition 4.6. *For every $\rho > 0$ there is $\tau > 0$ such that for every $(v_1, t_1), \dots, (v_k, t_k) \in \mathcal{C}(\eta)$ and every $\tau_1, \tau_2, \dots, \tau_k \in \mathbb{R}$ with the property that $\tau_{j+1} \geq \tau_j + t_j + \tau$ for all $1 \leq j < k$, there are $\tau'_j \in [\tau_j, \tau_j + \tau]$ and $w \in T^1 M$ such that $f_{\tau'_j}(w) \in B_{t_j}(v, \rho)$ for all $1 \leq j \leq k$.*

The additional ingredient in this statement over the specification property is that instead of asking that the transition times lie in the interval $[0, \tau]$, we can choose each transition time to be contained in a prescribed non-negative interval of length τ . To obtain this from the proof of Theorem 4.1, we modify the construction by gluing more than one copy of (v_0, t_0) between (v_{i-1}, t_{i-1}) and (v_i, t_i) ; we keep adding copies of (v_0, t_0) until the prescribed window of time for (v_i, t_i) is reached.

4.1. Closing lemma. Orbit segments in $\mathcal{C}(\eta)$ yield periodic orbits.

Lemma 4.7. *For all $\epsilon, \eta > 0$, there exists $T = T(\epsilon)$ so that for every $(v, t) \in \mathcal{C}(\eta)$, there are $w \in B_t(v, \epsilon)$ and $\tau \in [0, T]$ such that $f_{t+\tau} w = w$.*

Proof. We follow the proof of the Anosov closing lemma based on the Brouwer fixed point theorem. Without loss of generality, assume that $\epsilon > 0$ is small enough that for $(v, t) \in \mathcal{C}(\eta)$, the foliations W^u, W^{cs} have local product structure at $v, f_t v$ at scale ϵ with constant κ , and so do the foliations W^s, W^{cu} .

By Theorem 4.1, $\mathcal{C}(\eta)$ has specification at scale $\epsilon/4\kappa$; let T_0 be the transition time. Fix $(v_0, t_0) \in \mathcal{C}$ and $\alpha > 1$ such that $t_0 \geq 1$ and

$$(4.8) \quad d^s(f_{t_0} u, f_{t_0} u') \leq \alpha^{-1} d^s(u, u')$$

for all $u, u' \in W_\epsilon^s(v_0)$. Let $n \in \mathbb{N}$ satisfy $\alpha^n > 2\kappa$. By the specification property, there is a point w_0 whose forward orbit $\epsilon/4\kappa$ -shadows first (v, t) , then (v_0, t_0) , then (v_0, t_0) again, and so on until (v_0, t_0) has been shadowed n times, and then finally shadows (v, t) once more. In particular, we have $d_t(v, w_0) < \epsilon/4\kappa$, and there is $\tau \in [nt_0, n(t_0 + T_0) + T_0]$ such that $f_{t+\tau}(w_0) \in B(v, \epsilon/4\kappa)$, so $d_K(w_0, f_{t+\tau}w_0) < \epsilon/2\kappa$.

With w_0 fixed, consider the map $W_\epsilon^s(w_0) \rightarrow W_\epsilon^s(w_0)$ defined by $u \mapsto W_\epsilon^s(w_0) \cap W_\epsilon^{cu}(f_{t+\tau}u)$. This is well-defined because

$$\begin{aligned} d_K(f_{t+\tau}u, w_0) &\leq d_K(f_{t+\tau}u, f_{t+\tau}w_0) + d_K(f_{t+\tau}w_0, w_0) \\ &\leq \alpha^{-n}d_K(u, w_0) + \epsilon/2\kappa \leq \epsilon/\kappa, \end{aligned}$$

where the second inequality uses (4.8). By continuity of the map, the Brouwer fixed point theorem gives $w_1 \in W_\epsilon^s(w_0)$ with $w_1 \in W_\epsilon^{cu}(f_{t+\tau}w_1)$, and thus $w_1 \in W_\epsilon^u(f_{t+\tau+r}w_1)$ for some $|r| < \epsilon$. Using (4.4), we get

$$f_{-t-\tau-r}(W_{2\epsilon}^u(f_{t+\tau+r}w_1)) \subset W_{2\alpha^n\epsilon}^u(w_1) \subset W_\epsilon^u(w_1) \subset W_{2\epsilon}^u(f_{t+\tau+r}w_1),$$

and thus $f_{-t-\tau-r}$ sends $W_{2\epsilon}^u(f_{t+\tau+r}w_1)$ to itself continuously, so again the Brouwer fixed point theorem gives a fixed point w . This is periodic with period $t+\tau+r$, and $\tau+r \leq n(t_0+T_0)+T_0+\epsilon =: T$. Furthermore, since $d^u(w_1, s_0) \leq \epsilon$ and $d^{cs}(f_{t+1}w, f_{t+1}w_1) \leq 2\epsilon$, (4.2) and (4.3) give

$$\begin{aligned} d_t(v, w) &\leq d_t(v, w_0) + d_t(w_0, w_1) + d_t(w_1, w) \\ &< \epsilon/4\kappa + d^s(w_0, w_1) + d^u(f_{t+1}w_1, f_{t+1}w) \leq 4\epsilon, \end{aligned}$$

where the bound on $d_t(w_1, w)$ is because $f_{-\tau-r}w \in W_{2\epsilon}^u(f_t w_1)$, so $w = f_{-t-\tau-r}w \in B_t(w_1, 2\delta)$. \square

5. PRESSURE ESTIMATES

5.1. General estimates. We start with a general result for a continuous flow \mathcal{F} on a compact metric space X , which relates pressure for a collection of orbit segments to the free energies for an associated collection of measures. Given a collection \mathcal{C} of orbit segments, let $\mathcal{M}(\mathcal{C})$ denote the set of \mathcal{F} -invariant measures on X that are obtained as limits of convex combinations of empirical measures along orbit segments in \mathcal{C} . That is, for each $(x, t) \in \mathcal{C}$ define the empirical measure $\mathcal{E}_{x,t}$ by

$$\int \psi d\mathcal{E}_{x,t} = \int_0^t \psi(f_s x) ds,$$

for all $\psi \in C(X)$. Consider for each $t \geq 0$ the convex hull

$$\mathcal{M}_t(\mathcal{C}) = \left\{ \sum_{i=1}^k a_i \mathcal{E}_{x_i, t_i} : a_i \geq 0, \sum a_i = 1, (x_i, t_i) \in \mathcal{C} \right\}.$$

We use the following set of \mathcal{F} -invariant Borel probability measures:

$$(5.1) \quad \mathcal{M}(\mathcal{C}) = \left\{ \lim_{k \rightarrow \infty} \mu_{t_k} : t_k \rightarrow \infty, \mu_{t_k} \in \mathcal{M}_{t_k}(\mathcal{C}) \right\}.$$

Note that $\mathcal{M}(\mathcal{C})$ is non-empty as long as \mathcal{C} contains arbitrarily long orbit segments (which happens whenever $P(\mathcal{C}, \varphi) > -\infty$).

Proposition 5.1. *If φ is a continuous potential, then*

$$P(\mathcal{C}, \varphi) \leq \sup_{\mu \in \mathcal{M}(\mathcal{C})} P_\mu(\varphi),$$

where we write $P_\mu(\varphi) = h_\mu(\mathcal{F}) + \int \varphi d\mu$ for convenience.

Proof. For an arbitrary fixed $\epsilon > 0$, and any $t > 0$, let E_t be a (t, ϵ) -separated set for \mathcal{C}_t of maximal cardinality with

$$\log \sum_{y \in E_t} e^{\Phi(y, t)} > \log \Lambda(\mathcal{C}, \varphi, \epsilon, t) - 1.$$

Then there is $t_k \rightarrow \infty$ such that

$$(5.2) \quad \lim_{k \rightarrow \infty} \frac{1}{t_k} \sum_{y \in E_{t_k}} e^{\Phi(y, t_k)} \geq \lim_{k \rightarrow \infty} \frac{1}{t_k} (\log \Lambda(\mathcal{C}, \varphi, \epsilon, t_k) - 1) = P(\mathcal{C}, \varphi, \epsilon).$$

Consider the measures

$$\mu_t = \frac{\sum_{y \in E_t} e^{\Phi(y, t)} \mathcal{E}_{y, t}}{\sum_{y \in E_t} e^{\Phi(y, t)}}.$$

By passing to a subsequence if necessary, we can assume that $\mu_{t_k} \rightarrow \mu \in \mathcal{M}(\mathcal{C})$. The second half of the proof of the variational principle [22, Theorem 9.10] shows that $h(\mu) + \int \varphi d\mu \geq \liminf_{k \rightarrow \infty} \frac{1}{t_k} \sum_{y \in E_{t_k}} e^{\Phi(y, t_k)}$, so (5.2) gives $P_\mu(\varphi) \geq P(\mathcal{C}, \varphi, \epsilon)$. Taking $\epsilon > 0$ arbitrarily small gives the required result. \square

5.2. Pressure estimates for bad orbits. Now we consider the geodesic flow and estimate the pressure of the ‘bad’ orbit segments.

Proposition 5.2. *With $\mathcal{B}(\eta)$ as in §3.1 and $\varphi: T^1M \rightarrow \mathbb{R}$ continuous, we have $\lim_{\eta \rightarrow 0} P([\mathcal{B}(\eta)], \varphi) = P(\text{Sing}, \varphi)$. In particular, if $P(\text{Sing}, \varphi) < P(\varphi)$, then there exists some $\eta > 0$ such that $P([\mathcal{B}(\eta)], \varphi) < P(\varphi)$.*

Proof. Since the function λ vanishes on Sing , we have $\text{Sing} \times \mathbb{N} \subset [\mathcal{B}(\eta)]$ for all $\eta > 0$, which immediately gives $P(\text{Sing}, \varphi) \leq P([\mathcal{B}(\eta)], \varphi)$. Thus it suffices to show that for every $\epsilon > 0$ we have $P([\mathcal{B}(\eta)], \varphi) < P(\text{Sing}, \varphi) + \epsilon$ whenever $\eta > 0$ is sufficiently small.

To this end, consider for each $\eta > 0$ the set of measures $\mathcal{M}_\lambda(\eta) = \{\mu \in \mathcal{M}(T^1M) : \int \lambda d\mu \leq \eta\}$. Given $(v, t) \in [\mathcal{B}(\eta)]$, we have

$$\int_0^t \lambda(f_s v) ds \leq t\eta + 2\|\lambda\|,$$

where the last term comes from the fact that we are considering $[\mathcal{B}(\eta)]$ instead of $\mathcal{B}(\eta)$. By convexity, we have $\int \lambda d\mu_t \leq \eta + \frac{2}{t}\|\lambda\|$ for every $\eta_t \in \mathcal{M}_t([\mathcal{B}(\eta)])$, and thus every $\mu \in \mathcal{M}([\mathcal{B}(\eta)])$ satisfies $\int \lambda d\mu \leq \eta$, proving the inclusion $\mathcal{M}([\mathcal{B}(\eta)]) \subset \mathcal{M}_\lambda(\eta)$. By Proposition 5.1 we have

$$P([\mathcal{B}(\eta)], \varphi) \leq \sup_{\mu \in \mathcal{M}([\mathcal{B}(\eta)])} P_\mu(\varphi) \leq \sup_{\mu \in \mathcal{M}_\lambda(\eta)} P_\mu(\varphi),$$

and so it suffices to show that for every $\epsilon > 0$ this last quantity can be made smaller than $P(\text{Sing}, \varphi) + \epsilon$ by taking $\eta > 0$ sufficiently small.

Note that $\mathcal{M}_\lambda(\eta)$ is weak*-compact by continuity of λ . Moreover, $\mathcal{M}(\text{Sing}) \subset \mathcal{M}_\lambda(\eta)$ for all $\eta > 0$, and by Lemma 3.7, we see that every μ with $\int \lambda d\mu = 0$ is supported on Sing , whence we conclude that

$$(5.3) \quad \mathcal{M}(\text{Sing}) = \bigcap_{\eta > 0} \mathcal{M}_\lambda(\eta).$$

Let D be a metric on $\mathcal{M}(T^1M)$ compatible with the weak* topology. Since $\mathcal{M}_\lambda(\eta)$ is compact for each $\eta > 0$, (5.3) gives

$$D(\mathcal{M}_\lambda(\eta), \mathcal{M}(\text{Sing})) \rightarrow 0 \text{ as } \eta \rightarrow 0.$$

By [16, Proposition 3.3], f_t is h -expansive, so the entropy function $\mu \mapsto h(\mu)$ is upper semi-continuous, as is $\mu \mapsto P_\mu(\varphi)$. Thus, for any $\epsilon > 0$, there exists $\gamma > 0$ so that $D(\mu, \nu) < \gamma$ implies $P_\mu(\varphi) < P_\nu(\varphi) + \epsilon$. Choosing η small enough so that $D(\mathcal{M}_\lambda(\eta), \mathcal{M}(\text{Sing})) < \gamma$, we obtain

$$\sup_{\mu \in \mathcal{M}_\lambda(\eta)} P_\mu(\varphi) \leq \sup_{\mu \in \mathcal{M}(\text{Sing})} P_\mu(\varphi) + \epsilon = P(\text{Sing}, \varphi) + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this completes the proof. \square

5.3. Pressure of obstructions to expansivity. We now prove that $P_{\text{exp}}^\perp(\varphi) \leq P(\text{Sing}, \varphi)$. This is a corollary of the following lemma.

Lemma 5.3. *Suppose $\mu \in \mathcal{M}^e(\mathcal{F})$ satisfies $\mu(\text{NE}(\epsilon)) = 1$ for some $\epsilon > 0$. Then $\mu \in \mathcal{M}(\text{Sing})$.*

Proof. Given $v \in T^1M$ and $w \in \Gamma_\epsilon(v)$ and ϵ be sufficiently small. Suppose that γ_v and γ_w are different geodesics. Then by the flat strip theorem (see e.g. Proposition 1.11.4 of [10]), they bound a flat strip in the universal cover. Thus, v has a parallel Jacobi field, and hence $v \in \text{Sing}$. Thus, $\text{NE}(\epsilon) \subset \text{Sing}$. In particular, $\mu(\text{Sing}) = 1$. \square

As a corollary, we have the following.

Proposition 5.4. *For a continuous potential φ , $P_{\text{exp}}^\perp(\varphi) \leq P(\text{Sing}, \varphi)$.*

Proof. By Lemma 5.4 and the Variational Principle, for any $\epsilon > 0$,

$$\begin{aligned} P_{\text{exp}}^\perp(\varphi, \epsilon) &= \sup_{\mu \in \mathcal{M}^e(\mathcal{F})} \left\{ h_\mu(f_1) + \int \varphi d\mu : \mu(\text{NE}(\epsilon)) = 1 \right\} \\ &\leq \sup_{\mu \in \mathcal{M}(\text{Sing})} \left\{ h_\mu(f_1) + \int \varphi d\mu \right\} = P(\text{Sing}, \varphi). \quad \square \end{aligned}$$

6. COMPLETING THE PROOF OF THEOREM 3.1

Now we can apply Theorem 2.6, using the decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S}) = (\mathcal{B}(\eta), \mathcal{G}(\eta), \mathcal{B}(\eta))$ described in §3.1. By Proposition 4.1, $\mathcal{G}(\eta)$ has specification all η . By Proposition 5.2, for sufficiently small η we have $P([\mathcal{P}] \cup [\mathcal{S}], \varphi) = P([\mathcal{B}(\eta)], \varphi) < P(\varphi)$. By Proposition 5.4, $P_{\text{exp}}^\perp(\varphi) \leq P(\text{Sing}, \varphi)$. This verifies the hypotheses of Theorem 2.6, and thus we conclude that φ has a unique equilibrium state μ .

To prove the remaining properties of μ stated in Theorem 3.1, we start by observing that μ is ergodic, and thus either $\mu(\text{Sing}) = 0$ or $\mu(\text{Sing}) = 1$. Suppose the second case holds. Then by the variational principle, it would follow that $P(\text{Sing}, \varphi) \geq h_\mu(\mathcal{F}) + \int \varphi d\mu = P(\varphi)$. This contradicts the hypothesis of the theorem, and thus $\mu(\text{Reg}) = 1$.

To prove μ is fully supported, and that weighted regular periodic orbits equidistribute to μ , we recall details from [6]. Given a decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ and $M > 0$, we write \mathcal{G}^M for the set of orbit segments (x, t) whose decomposition satisfies $p(x, t), s(x, t) \leq M$. When the hypotheses of Theorem 2.6 are satisfied, \mathcal{G}^M has the following properties.

Lemma 6.1. *For sufficiently large M , we have $P(\mathcal{G}^M, \varphi) = P(\varphi)$. We have the lower Gibbs property on \mathcal{G}^M : for all $\rho > 0$, there exists $Q, T, M > 0$ such that for every $(v, t) \in \mathcal{G}^M$ with $t \geq T$,*

$$\mu(B_t(v, \rho)) \geq Qe^{-tP(\varphi) + \Phi(v, t)}.$$

As a consequence of the lower Gibbs property, if $(v, t) \in \mathcal{G}$ and t is sufficiently large, then $\mu(B(v, \rho)) > 0$.

Proof. For sufficiently large M , [6, Lemma 4.12] shows that $P(\mathcal{G}^M, \varphi) = P(\varphi)$. The lower Gibbs property for \mathcal{G}^M is provided by [6, Lemma 4.16]. That lemma also involves a scale $\delta < \rho/2$, at which \mathcal{G} is required have specification, and at which the pressure gap holds, see [6, Remark 4.13]; both of these conditions hold here for arbitrarily small δ , so [6, Lemma 4.16] applies. Finally, note that $\mathcal{G} \subset \mathcal{G}^M$ for all M . Thus, if $(v, t) \in \mathcal{G}$ and $t \geq T$, then $\mu(B(v, \rho)) \geq \mu(B_t(v, \rho)) \geq Qe^{-tP(\varphi) + \Phi(v, t)} > 0$. \square

We also need the following consequence of Theorem 4.1.

Lemma 6.2. *Given $\eta, \rho > 0$, there is $\eta_0 > 0$ such that for every $v \in \text{Reg}(\eta)$ and every $T > 0$, there are $t \geq T$ and $w \in B(v, \rho)$ such that $(w, t) \in \mathcal{G}(\eta_0)$.*

Proof. By Lemma 3.9, we can decrease ρ if necessary and assume that if $(u, t) \in \mathcal{G}(\eta/2)$ and $u' \in B_t(u, \rho)$, then $u' \in \mathcal{G}(\eta/4)$. Let τ be the transition time for the specification property for $\mathcal{G}(\eta/2)$ at scale ρ . Let $v \in \text{Reg}(\eta)$. Then using the modulus of continuity for λ , we can find a fixed $\epsilon > 0$ (independent of v) so that $(v, \epsilon) \in \mathcal{G}(\eta/2)$. Fix $(u, t_0) \in \mathcal{G}(\eta/2)$, and let $k \in \mathbb{N}$ be such that $kt_0 \geq T$. By the specification property, we can find a point w that shadows (v, ϵ) , and then shadows k copies of (u, t_0) . Then for each $j \geq 1$, $(f_{s_j + \tau_j} w, t_0) \in \mathcal{G}(\eta/4)$. Using this fact, and the definition of \mathcal{G} , it is not hard to show the existence of a constant η_0 so that $(w, t) \in \mathcal{G}(\eta_0)$ where $t = kt_0 + \epsilon + \sum_{j=1}^{k-1} \tau_j$, and η_0 depends only on ρ, η, τ . \square

We are now ready to prove the following.

Proposition 6.3. *The unique equilibrium state μ provided by Theorem 3.1 is fully supported.*

Proof. We exhibit a dense set Z such that for every $v \in Z$ and $\rho > 0$ we have $\mu(B(v, 2\rho)) > 0$. Take $Z = \{v : \lambda(v) > 0\}$. By Lemma 3.11, Z is dense, and by Lemma 6.2, for every $v \in Z$ and $\rho > 0$ there exists $\eta_0 > 0$ such that for every $T > 0$, there are $t \geq T$ and $w \in B(v, \rho)$ such that $(w, t) \in \mathcal{G}(\eta_0)$. The decomposition $(\mathcal{B}(\eta_0), \mathcal{G}(\eta_0), \mathcal{B}(\eta_0))$ satisfies the conditions of Theorem 2.6, and so Lemma 6.1 applies. We are free to assume that (w, t) is chosen with t as large as we like, so Lemma 6.1 shows that $\mu(B(v, 2\rho)) \geq \mu(B(w, \rho)) > 0$. \square

Proposition 6.4. *The unique equilibrium measure μ is the weak* limit of weighted regular periodic orbit measures.*

Proof. By the discussion in §2.3, it suffices to prove $P_{\text{Reg}}^*(\varphi) = P(\varphi)$. Lemma 6.1 gives M so that $P(\mathcal{G}^M, \varphi) = P(\varphi)$. Given $\epsilon > 0$, by continuity of the flow, there exists $\epsilon' > 0$ such that $d_K(v, w) < \epsilon'$ implies that $d_K(f_t v, f_t w) < \delta$ for every $t \in [-M, M]$. Let $(v, t) \in \mathcal{G}^M$ with $t > 2M$. We show that (v, t) can be ϵ -shadowed by a regular periodic orbit. There exists $p, s \leq M$ so that $(v', t') = (f_p v, t - p - s) \in \mathcal{G}$. By Lemma 4.7, we know that there exists w with $f_{\nu + \tau} w = w$, where $\tau \in [0, T(\epsilon')]$, and $d_{\nu'}(v', w) < \epsilon'$. It follows that $d_t(v, f_{-p} w) < \epsilon$, and thus (v, t) can be ϵ' -shadowed by a regular periodic orbit whose length is at most $t + T(\epsilon')$. It follows that $P_{\text{Reg}}^*(\varphi) \geq P(\mathcal{G}^M, \varphi) = P(\varphi)$. \square

7. THE BOWEN PROPERTY

We show that Hölder continuous potentials on T^1M have the Bowen property on $\mathcal{G}(\eta)$. Then we show that the geometric potential has the Bowen property on $\mathcal{G}(\eta)$, despite the fact that it is not known whether this potential is Hölder continuous. It is immediate from these results that any potential of the form $p\varphi + q\varphi^u$, where φ is Hölder and $p, q \in \mathbb{R}$, has the Bowen property.

7.1. Hölder continuous potentials. We start by working along stable and unstable leaves, then use the local product structure.

Definition 7.1. *A potential $\varphi: T^1M \rightarrow \mathbb{R}$ is Hölder along stable leaves if there are $C, \theta, \epsilon > 0$ such that for any $v \in T^1M$ and $w \in W_\epsilon^s(v)$, we have $|\varphi(v) - \varphi(w)| \leq Cd^s(v, w)^\theta$. Similarly, φ is Hölder along unstable leaves if there are $C, \theta, \epsilon > 0$ such that $|\varphi(v) - \varphi(w)| \leq Cd^u(v, w)^\theta$ whenever $v \in T^1M$ and $w \in W_\epsilon^u(v)$.*

By (4.1) and (4.3), which bound d_K in terms of d^u and d^s , a Hölder continuous potential is Hölder along both stable and unstable leaves.

Definition 7.2. *A potential φ has the Bowen property along stable leaves with respect to $\mathcal{C} \subset T^1M \times [0, \infty)$ if there are $\delta, K > 0$ such that*

$$\sup\{|\Phi(v, t) - \Phi(w, t)| : (v, t) \in \mathcal{C}, w \in W_\delta^s(v)\} \leq K.$$

A potential φ has the Bowen property along unstable leaves with respect to \mathcal{C} if there are $\delta, K > 0$ such that

$$\sup\{|\Phi(v, t) - \Phi(w, t)| : (v, t) \in \mathcal{C}, w \in f_{-t}W_\delta^u(f_tv)\} \leq K.$$

Lemma 7.3. *If φ is Hölder along stable leaves (respectively unstable leaves), then it has the Bowen property along stable leaves (respectively unstable leaves) with respect to $\mathcal{G}(\eta)$ for any $\eta > 0$.*

Proof. We give the proof for stable leaves; the unstable case is similar. Let $\delta > 0$ be as in Lemma 3.9. Let $(v, T) \in \mathcal{G}(\eta)$ and $w \in W_\delta^s(v)$. By Lemma 3.9 and the Hölder property along stable leaves, we have $|\varphi(f_tv) - \varphi(f_tw)| \leq Ce^{-\frac{\eta}{2}\theta t}$ for each $t \in [0, T]$. Thus, we have

$$|\Phi(v, T) - \Phi(w, T)| \leq C \int_0^T e^{-\frac{\eta}{2}\theta t} dt \leq C \int_0^\infty e^{-\frac{\eta}{2}\theta t} dt.$$

This bound is independent of v and T , which proves the lemma. \square

Lemma 7.4. *Given $\eta > 0$, suppose that $\varphi: T^1M \rightarrow \mathbb{R}$ has the Bowen property on $\mathcal{G}(\eta/2)$ with respect to both stable and unstable leaves. Then φ has the Bowen property on $\mathcal{G}(\eta)$.*

Proof. Since curvature of horospheres is uniformly bounded on T^1M , there are $\delta_0, C > 0$ such that for every $v \in T^1M$ and $w \in W^u(v)$ with $d^u(v, w) \leq \delta_0$, we have $d^u(v, w) \leq Cd_K(v, w)$. Let $\delta_1 > 0$ be such that for every $(v, T) \in \mathcal{G}(\eta)$, the foliations W^u, W^{cs} have local product structure at scale δ_1 with constant κ at both v and $f_T v$. By Lemma 3.9, there exists $\delta_2 > 0$ so that for $(v, T) \in \mathcal{G}(\eta)$, every $w \in B_T(v, \delta_2)$ has $(w, T) \in \mathcal{G}(\eta/2)$. Let $\delta_3, K > 0$ be the constants associated to the Bowen property for ϕ with respect to $\mathcal{G}(\eta/2)$ along stable and unstable leaves, and assume without loss of generality that $\delta_3 < \delta_0$.

Now take $0 < \delta < \min(\delta_0, \delta_1, \delta_2, \delta_3/(2\kappa C))$. Fix $(v, T) \in \mathcal{G}(\eta)$ and $w \in B_T(v, \delta)$. By LPS, there is $v' \in W_{\delta\kappa}^{cs}(v) \cap W_{\delta\kappa}^u(w)$. We claim that $f_T v' \in W_{\delta_3}^u(f_T w)$. Suppose this fails; then there is $t \in [0, T]$ such that

$$(7.1) \quad \delta_3 < d^u(f_t v', f_t w) \leq \delta_0$$

but since $v' \in W_{\delta\kappa}^{cs}(v) \subset B_T(v, \delta\kappa)$, we have

$$d_K(f_t v', f_t w) \leq d_K(f_t v', f_t v) + d_K(f_t v, f_t w) \leq 2\delta\kappa,$$

and so $d^u(f_t v', f_t w) \leq 2\delta\kappa C < \delta_3$, contradicting (7.1). It follows that $v' \in f_{-T}W_{\delta_3}^s(w)$. Let $\rho \in [-\kappa\delta, \kappa\delta]$ be such that $f_\rho(v') \in W_{\delta_3}^s(v)$; then

$$\begin{aligned} |\Phi(v, T) - \Phi(w, T)| &\leq |\Phi(v, T) - \Phi(f_\rho v', T)| + |\Phi(f_\rho v', T) - \Phi(v', T)| \\ &\quad + |\Phi(v', T) - \Phi(w, T)| \leq K + 2\kappa\delta\|\varphi\| + K. \quad \square \end{aligned}$$

The following is an immediate consequence of Lemmas 7.3 and 7.4.

Corollary 7.5. *If φ is Hölder continuous, then it has the Bowen property with respect to $\mathcal{G}(\eta)$ for any $\eta > 0$.*

7.2. The geometric potential. The *geometric potential* for geodesic flow is given by

$$\varphi^u(v) = -\lim_{t \rightarrow 0} \frac{1}{t} \log \det(df_t|_{E_v^u}) = -\frac{d}{dt} \Big|_{t=0} \log \det(df_t|_{E_v^u}).$$

When M has dimension 2, the function φ^u is Hölder along unstable leaves [14, Proposition III], and so the problem of proving the Bowen property for φ^u on $\mathcal{G}(\eta)$ reduces to proving it along stable leaves, where it is not known whether φ^u is Hölder. In higher dimensions, it is not known whether φ^u is Hölder continuous on either stable or unstable leaves; an advantage of our approach is that we sidestep the question of Hölder regularity by proving the Bowen property on \mathcal{G} directly.

We will find it more convenient to work with the potential function

$$(7.2) \quad \psi^u(v) = -\lim_{t \rightarrow 0} \frac{1}{t} \log \det(J_{v,t}^u) = -\frac{d}{dt} \Big|_{t=0} \log \det(J_{v,t}^u),$$

where $J_{v,t}^u: v^\perp \rightarrow (f_tv)^\perp$ is the linear map that takes $w \in v^\perp$ to the value at t of the unstable Jacobi field along γ_v that has value w at 0.

Lemma 7.6. *There exists K so that $|\int_0^T \varphi^u(f_tv) dt - \int_0^T \psi^u(f_tv) dt| \leq K$ for all $v \in T^1M$ and $T > 0$.*

Proof. Given $v \in T^1M$, let ω be the volume form on $E_v^u \subset T_vT^1M$ induced by the Sasaki metric, and let ω' be the volume form on $v^\perp \subset T_{\pi v}M$ induced by the Riemannian metric. The canonical projection $\pi: T^1M \rightarrow M$ has derivative $d\pi: TT^1M \rightarrow TM$ that takes E_v^u to v^\perp and pushes forward ω to a volume form $(d\pi)_*\omega$ on v^\perp ; this need not be the same volume form as ω' , but by considering each volume form as a wedge product of 1-forms associated to unit vectors, it follows from (2.10) that there is $C > 0$ such that $\omega'_{\pi v} \leq (d\pi)_*\omega_v \leq C\omega'_{\pi v}$ for every v . The lemma follows since $\int_0^T \varphi^u(f_tv) dt = \log((df_t)_*\omega_v/\omega_{f_tv})$, and similarly for ψ^u and ω' . \square

It follows that $q\varphi^u$ and $q\psi^u$ share the same equilibrium states for any $q \in \mathbb{R}$, and that $q\varphi^u$ has the Bowen property on $\mathcal{G}(\eta)$ if and only if ψ^u does. From now on, we work with ψ^u . Let $\mathcal{U}_v^u(t)$ be the second fundamental form of the unstable horosphere $H^u(f_tv)$, as in §2.4.2, so $\mathcal{U}_v^u(t)$ is a positive semidefinite symmetric linear operator on $(f_tv)^\perp$ such that if $J(t)$ is an unstable Jacobi field along γ_v , then $J'(t) = \mathcal{U}_v^u(t)J(t)$; see Lemma 2.7. Now (7.2) gives $\psi^u(v) = -\text{tr} \mathcal{U}_v^u(0)$, so

$$(7.3) \quad \int_0^T \psi^u(f_tv) dt = - \int_0^T \text{tr} \mathcal{U}_v^u(t) dt.$$

The rest of this section is devoted to proving the following.

Proposition 7.7. *For every $\eta > 0$ there are $\delta, Q, \xi > 0$ such that given any $(v, T) \in \mathcal{G}(\eta)$, $w \in W_\delta^s(v)$, and $w' \in f_{-T}W_\delta^u(f_Tv)$, for every $0 \leq t \leq T$ we have*

$$(7.4) \quad |\text{tr} \mathcal{U}_v^u(t) - \text{tr} \mathcal{U}_w^u(t)| \leq Qe^{-\xi t},$$

$$(7.5) \quad |\text{tr} \mathcal{U}_v^u(t) - \text{tr} \mathcal{U}_{w'}^u(t)| \leq Q(e^{-\xi t} + e^{-\xi(T-t)}).$$

Since $\int_0^T \psi^u(f_tv) dt = - \int_0^T \text{tr} \mathcal{U}_v^u(t) dt$, (7.4) shows that ψ^u has the Bowen property on $\mathcal{G}(\eta)$ along stable leaves, and (7.5) gives it along unstable leaves. Thus, by Lemma 7.4, ψ^u has the Bowen property on $\mathcal{G}(2\eta)$, so we obtain the desired result:

Corollary 7.8. *For every $\eta > 0$, the potential ψ^u , and thus the potential φ^u , has the Bowen property on $\mathcal{G}(\eta)$.*

To prove Proposition 7.7, we study $\mathcal{U}_v^u(t)$ by using the fact that its time evolution is governed by a Riccati equation, which we now describe. For $v \in T^1M$, let $\mathcal{K}(v): v^\perp \rightarrow v^\perp$ be the symmetric linear map such that $\langle \mathcal{K}(v)X, Y \rangle = \langle R(X, v)v, Y \rangle$ for $X, Y \in v^\perp$. The eigenvalues of $\mathcal{K}(v)$ are sectional curvatures of planes containing v . Consequently $\mathcal{K}(v)$ is negative semidefinite. Recalling (2.5), we see that Jacobi fields along γ_v evolve according to $J''(t) + \mathcal{K}(f_tv)J(t) = 0$.

Lemma 2.7 shows that if $J(t)$ arises from varying $\gamma = \gamma_v$ through unit speed geodesics orthogonal to a hypersurface H , then $J'(t) = \mathcal{U}(t)J(t)$, where $\mathcal{U}(t)$ is the second fundamental form of f_tH . Differentiating this, the second-order ODE above becomes

$$0 = J''(t) + \mathcal{K}(\dot{\gamma}(t))J(t) = (\mathcal{U}'(t) + \mathcal{U}^2(t) + \mathcal{K}(\dot{\gamma}(t)))J(t).$$

This shows that $\mathcal{U}(t)$ is a solution of the Riccati equation along γ :

$$(7.6) \quad \mathcal{U}'(t) + \mathcal{U}^2(t) + \mathcal{K}(\dot{\gamma}(t)) = 0.$$

Using parallel translation along γ to identify the spaces $\dot{\gamma}(t)^\perp$, we can represent \mathcal{U} and \mathcal{K} by symmetric $(n-1) \times (n-1)$ matrices. Note that $\mathcal{U}_{\dot{\gamma}(0)}^u(t)$ is the unique solution of (7.6) that is positive semidefinite for all $t \in \mathbb{R}$ and bounded for $t \leq 0$.

When M is a surface, the Riccati equation (7.6) along γ_v becomes

$$(7.7) \quad U'(t) + U^2(t) + K(f_tv) = 0,$$

where $K(f_tv)$ is the Gaussian curvature at $\gamma_v(t)$. A nice exposition of the Riccati equation for non-positive curvature surfaces is in [19].

We now prove Proposition 7.7. Let V be the space of symmetric $(n-1) \times (n-1)$ matrices, equipped with the semi-metric

$$\rho(\mathcal{A}, \mathcal{B}) = |\operatorname{tr} \mathcal{A} - \operatorname{tr} \mathcal{B}|.$$

Given $v \in T^1M$ and $s \leq t \in \mathbb{R}$, let $\mathcal{R}_{s,t}^v: V \rightarrow V$ denote the time-evolution map from time s to time t for the nonautonomous ODE

$$(7.8) \quad \mathcal{U}'(\tau) + \mathcal{U}^2(\tau) + \mathcal{K}(f_\tau v) = 0.$$

That is, $\mathcal{R}_{s,t}^v(\mathcal{U}_0) = \mathcal{U}(t)$, where \mathcal{U} is the solution of (7.8) with $\mathcal{U}(s) = \mathcal{U}_0$. Then given $v, w \in T^1M$, we have

$$(7.9) \quad \begin{aligned} \rho(\mathcal{U}_v^u(t), \mathcal{U}_w^u(t)) &= \rho(\mathcal{R}_{0,t}^v \mathcal{U}_v^u(0), \mathcal{R}_{0,t}^w \mathcal{U}_w^u(0)) \\ &\leq \rho(\mathcal{R}_{0,t}^v \mathcal{U}_v^u(0), \mathcal{R}_{0,t}^v \mathcal{U}_w^u(0)) + \rho(\mathcal{R}_{0,t}^v \mathcal{U}_w^u(0), \mathcal{R}_{0,t}^w \mathcal{U}_w^u(0)). \end{aligned}$$

To estimate the first term, we will establish contraction properties of $\mathcal{R}_{0,t}^v$ on a suitable subset of V . Given $\mathcal{A}, \mathcal{B} \in V$, write $\mathcal{A} \succcurlyeq \mathcal{B}$ if $\mathcal{A} - \mathcal{B}$ is positive semi-definite and $\mathcal{A} \succ \mathcal{B}$ if $\mathcal{A} - \mathcal{B}$ is positive definite. Similarly, write $\mathcal{A} \preccurlyeq \mathcal{B}$ if $\mathcal{A} - \mathcal{B}$ is negative semi-definite and $\mathcal{A} \prec \mathcal{B}$ if $\mathcal{A} - \mathcal{B}$ is negative definite. Fix $b > 0$ such that $-b^2$ is a strict lower bound for

the sectional curvatures of M , and let $D = \{\mathcal{U} \in V : 0 \preceq \mathcal{U} \preceq b\mathcal{I}\}$. The following lemma, proved in §7.3, shows that D is a forward-invariant domain for the maps $\mathcal{R}_{s,t}^v$.

Lemma 7.9. *For every $v \in T^1M$ and $s \leq t \in \mathbb{R}$, we have $\mathcal{R}_{s,t}^v D \subset D$.*

Henceforth, we use the letter Q generically for a constant whose precise value will be different at different occurrences. Recall that the function $\tilde{\lambda} \geq 0$ was defined in §3.3 as $\tilde{\lambda}(v) = \max(0, \lambda(v) - \frac{\eta}{2})$. The following lemma allows us to estimate the first term in (7.9).

Lemma 7.10. *For every $\eta > 0$, there is a constant $Q > 0$ such that for every $v \in T^1M$, $s \leq t \in \mathbb{R}$, and $\mathcal{U}_0, \mathcal{U}_1 \in D$, we have*

$$(7.10) \quad \rho(\mathcal{R}_{s,t}^v \mathcal{U}_0, \mathcal{R}_{s,t}^v \mathcal{U}_1) \leq Q e^{-\int_s^t \tilde{\lambda}(f_\tau v) d\tau} \|\mathcal{U}_0 - \mathcal{U}_1\|.$$

We prove Lemma 7.10 in §7.4. To estimate the second term in (7.9), we fix $v, w \in T^1M$, $t \geq 0$, and $\mathcal{U}_0 \in D$, and consider the function $R = R_{\mathcal{U}_0}^{v,w,t} : [0, t] \rightarrow D$ given by

$$(7.11) \quad R(s) = \mathcal{R}_{s,t}^v \mathcal{R}_{0,s}^w \mathcal{U}_0,$$

so $R(s)$ evolves \mathcal{U}_0 by the Ricatti equation for w until time s , then evolves by the Ricatti equation for v from time s to time t . Our proof of Lemma 7.10 shows that $\mathcal{U}_w^u(0) \in D$, so we can set $\mathcal{U}_0 = \mathcal{U}_w^u(0)$ to obtain a path in D that connects $R(0) = \mathcal{R}_{0,t}^v \mathcal{U}_w^u(0)$ to $R(t) = \mathcal{R}_{0,t}^w \mathcal{U}_w^u(0)$. Thus we can estimate the second term in (7.9) by bounding the length of the path R in the pseudo-metric ρ .

Lemma 7.11. *Given any $v, w \in T^1M$ and $t \geq 0$, the function $R = R_{\mathcal{U}_w^u(0)}^{v,w,t}$ satisfies the following bound for all $0 \leq s_1 \leq s_2 \leq t$:*

$$(7.12) \quad \rho(R(s_1), R(s_2)) \leq \int_{s_1}^{s_2} Q e^{-\int_s^t \tilde{\lambda}(f_\tau v) d\tau} \|\mathcal{K}(f_s v) - \mathcal{K}(f_s w)\| ds.$$

We prove Lemma 7.11 in §7.5. We now explain how to prove Proposition 7.7 from Lemmas 7.10 and 7.11. Given $\eta > 0$, let $\delta > 0$ be as in (3.1). Given $(v, T) \in \mathcal{G}$ and $w \in W_\delta^s(v)$, smoothness of $\mathcal{K} : T^1M \rightarrow V$ together with (3.3) gives

$$\|\mathcal{K}(f_s v) - \mathcal{K}(f_s w)\| \leq Q d_K(f_s v, f_s w) \leq Q d^s(f_s v, f_s w) \leq Q \delta e^{-\int_0^s \tilde{\lambda}(f_\tau v) d\tau}$$

for all $s \in [0, T]$. We conclude that for every $t \in [0, T]$, the integrand in (7.12) is bounded above by

$$Q e^{-\int_0^t \tilde{\lambda}(f_\tau v) dt} \leq Q e^{-\int_0^t \lambda(f_\tau v) dt + \frac{\eta}{2}t} \leq Q e^{-\frac{\eta}{2}t},$$

where the last inequality holds because $(v, T) \in \mathcal{G}(\eta)$. Thus, (7.12) gives $\rho(R(s_1), R(s_2)) \leq (s_2 - s_1)Qe^{-\frac{\eta}{2}t}$. Fixing $\xi < \frac{\eta}{2}$, and setting $s_1 = 0$, $s_2 = t$, we obtain

$$\rho(\mathcal{R}_{0,t}^v \mathcal{U}_w^u(0), \mathcal{R}_{0,t}^w \mathcal{U}_w^u(0)) \leq Qte^{-\frac{\eta}{2}t} < Qe^{-\xi t},$$

which bounds the second term of (7.9). By (7.10), we have

$$(7.13) \quad \rho(\mathcal{R}_{0,t}^v \mathcal{U}_v^u(0), \mathcal{R}_{0,t}^v \mathcal{U}_w^u(0)) \leq Qe^{-\int_0^t \lambda(f_\tau v) d\tau} \leq Qe^{-\eta t},$$

which bounds the first term of (7.9). Thus, both terms of (7.9) are bounded above by $Qe^{-\xi t}$, which proves the first half of Proposition 7.7.

To prove (7.5), first observe that when $(v, T) \in \mathcal{G}$ and $f_T w' \in W_\delta^u(f_T v)$, we can use (3.6) to get

$$\|\mathcal{K}(f_s v) - \mathcal{K}(f_s w')\| \leq Qe^{-\int_s^T \tilde{\lambda}(f_t w') dt} \leq Qe^{-\frac{\eta}{2}(T-s)}.$$

Now letting $R = R_{\mathcal{U}_w^u(0)}^{v, w', t}$ and $t \in [0, T]$, (7.12) gives the bound

$$\begin{aligned} \rho(R(0), R(t)) &\leq Q \int_0^t \|\mathcal{K}(f_s v) - \mathcal{K}(f_s w')\| ds \\ &\leq Q \int_0^t e^{-\frac{\eta}{2}(T-s)} ds \leq Qe^{-\frac{\eta}{2}(T-t)}. \end{aligned}$$

Thus, $\rho(\mathcal{R}_{0,t}^v \mathcal{U}_{w'}^u(0), \mathcal{R}_{0,t}^{w'} \mathcal{U}_w^u(0)) \leq Qe^{-\frac{\eta}{2}(T-t)}$. Also, (7.13) holds with w' in place of w . Using these bounds in (7.9) gives (7.5) with $\xi = \frac{\eta}{2}$. Modulo the proofs of Lemmas 7.9, 7.10 and 7.11, which are given in the next sections, this completes the proof of Proposition 7.7.

7.3. Proof of Lemma 7.9. The following three lemmas give forward-invariance of the domain D under the maps $\mathcal{R}_{s,t}^v$ for any $v \in T^1 M$.

Lemma 7.12. [7, p. 50] *Suppose $\mathcal{U}_1(t)$ and $\mathcal{U}_2(t)$ are symmetric solutions of (7.6) with $\mathcal{U}_1(t_0) \succcurlyeq \mathcal{U}_2(t_0)$. Then $\mathcal{U}_1(t) \succcurlyeq \mathcal{U}_2(t)$ for all t . Similarly, if $\mathcal{U}_1(t_0) \succ \mathcal{U}_2(t_0)$, then $\mathcal{U}_1(t) \succ \mathcal{U}_2(t)$ for all t .*

Proof. Both $\mathcal{D}(t) = \mathcal{U}_1(t) - \mathcal{U}_2(t)$ and $\mathcal{M}(t) = \frac{1}{2}(\mathcal{U}_1(t) + \mathcal{U}_2(t))$ are symmetric and by a straightforward computation, satisfy

$$\mathcal{D}' + \mathcal{D}\mathcal{M} + \mathcal{M}\mathcal{D} = 0.$$

Let $\mathcal{X}(t)$ be the solution of $\mathcal{X}'(t) = \mathcal{M}(t)\mathcal{X}(t)$ with $\mathcal{X}(t_0) = \mathcal{I}$. Then $\mathcal{X}(t)$ is non-singular for all t and, since \mathcal{M} is symmetric,

$$(\mathcal{X}^* \mathcal{D} \mathcal{X})' = \mathcal{X}^* (\mathcal{D}' + \mathcal{D}\mathcal{M} + \mathcal{M}\mathcal{D}) \mathcal{X} = 0.$$

Thus $\mathcal{X}^* \mathcal{D} \mathcal{X}(t)$ is constant, so the signature of $\mathcal{D}(t)$ is constant. \square

Lemma 7.13. *Let $\mathcal{U}(t)$ be a symmetric solution of (7.6) with $\mathcal{U}(t_0) \succcurlyeq 0$. Then $\mathcal{U}(t) \succcurlyeq 0$ for all $t \geq t_0$.*

Proof. Let $\mathcal{U}_\epsilon(t)$ be the (symmetric) solution of

$$(7.14) \quad \mathcal{U}'_\epsilon(t) + \mathcal{U}_\epsilon^2(t) + \mathcal{K}(\dot{\gamma}(t)) - \epsilon^2 \mathcal{I} = 0$$

with $\mathcal{U}_\epsilon(t_0) = \mathcal{U}(t_0) \succcurlyeq 0$. Then $\lim_{\epsilon \rightarrow 0} \mathcal{U}_\epsilon(t) = \mathcal{U}(t)$ for all t , so it suffices to prove that $\mathcal{U}_\epsilon(t) \succcurlyeq 0$ for all $t \geq t_0$ and $\epsilon > 0$. Let

$$S = \{t \geq t_0 \mid \mathcal{U}_\epsilon(t_1) \succcurlyeq 0 \text{ for all } t_1 \in [t_0, t]\}.$$

Suppose S is bounded above, and let $t_1 = \sup S$. Let \mathcal{U}_ϵ^1 be the solution of (7.14) with $\mathcal{U}_\epsilon^1(t_1) = 0$. By Lemma 7.12, we have $\mathcal{U}_\epsilon(t) \succcurlyeq \mathcal{U}_\epsilon^1(t)$ for all $t \in \mathbb{R}$. However, $(\mathcal{U}_\epsilon^1)'(t_1) = -\mathcal{K}(\dot{\gamma}(t_1)) + \epsilon^2 \mathcal{I}$ is positive definite, so there is some $t_2 > t_1$ with the property that $\mathcal{U}_\epsilon^1(t) \succ 0$ for all $t \in (t_1, t_2]$, and consequently $\mathcal{U}_\epsilon(t) \succcurlyeq 0$ for all $t \in (t_1, t_2]$. This means that $t_2 \in S$, contradicting maximality of t_1 . We conclude that $S = [t_0, \infty)$, which proves the lemma. \square

Recall that $b > 0$ was chosen so that $-b^2$ is a strict lower bound for the sectional curvatures of M .

Lemma 7.14. *Suppose $\mathcal{U}(t)$ is a solution of (7.6) with $b\mathcal{I} \succcurlyeq \mathcal{U}(t_0)$. Then $b\mathcal{I} \succcurlyeq \mathcal{U}(t)$ for $t \geq t_0$.*

Proof. Proceed as in Lemma 7.13 by observing that $\mathcal{U}'(t) = -\mathcal{U}^2(t) - \mathcal{K}(\dot{\gamma}(t)) \prec 0$ if $\mathcal{U}(t) = b\mathcal{I}$ and applying Lemma 7.12. \square

We conclude that D is an invariant domain for evolution under the Riccati equation (7.6). Thus, for every $v \in T^1M$ and $s \leq t \in \mathbb{R}$, we have $\mathcal{R}_{s,t}^v D \subset D$.

7.4. Proof of Lemma 7.10. We begin by proving convergence results to \mathcal{U}^u for Riccati solutions with positive semi-definite initial conditions.

Lemma 7.15. *Let $\mathcal{U}_{v,\tau}^u$ be the solution of the Riccati equation along γ_v such that $\mathcal{U}_{v,\tau}^u(-\tau) = 0$. Then $\mathcal{U}_{v,\tau}^u(0) \rightarrow \mathcal{U}_v^u(0)$ as $\tau \rightarrow \infty$. The convergence is uniform in v .*

Proof. We have $\mathcal{U}_{v,\tau}^u(-\tau) = 0 \preccurlyeq \mathcal{U}_v^u(f_{-\tau}v) = \mathcal{U}_v^u(-\tau)$. It follows from Lemma 7.12 that $\mathcal{U}_{v,\tau}^u(t) \preccurlyeq \mathcal{U}_v^u(t)$ for all t , in particular when $t = 0$. On the other hand, Lemma 7.13 tells us that $\mathcal{U}_{v,\tau}^u(t) \succcurlyeq 0$ for $t \geq -\tau$. It follows that if $0 \leq \tau_1 \leq \tau_2$, then $0 \preccurlyeq \mathcal{U}_{v,\tau_1}^u(0) \preccurlyeq \mathcal{U}_{v,\tau_2}^u(0) \preccurlyeq \mathcal{U}_v^u(0)$. We would like to deduce that $\mathcal{U}_{v,\tau}^u(0)$ converges to $\mathcal{U}_v^u(0)$ as $\tau \rightarrow \infty$.

Observe that for every $x \in \mathbb{R}^{n-1}$, the sequence $\langle x, \mathcal{U}_{v,\tau}^u(0)x \rangle$ is monotonic in τ , and hence has a limit as $\tau \rightarrow \infty$. Since this holds for every x , we conclude that $\lim_{\tau \rightarrow \infty} \mathcal{U}_{v,\tau}^u(0)$ exists and that it is $\preccurlyeq \mathcal{U}_v^u(0)$; it remains to show that the limit is in fact $\mathcal{U}_v^u(0)$ for each v .

Let $J_{v,w,\tau}$ be a Jacobi field along γ_v such that $J_{v,w,\tau}(0) = w \in v^\perp$ and $J'_{v,w,\tau}(-\tau) = 0$. Since the norm of a Jacobi field is a convex function,

we have $\|J_{v,w,\tau}(t)\| \leq \|w\|$ for $-\tau \leq t \leq 0$. If τ_k is a sequence such that $\tau_k \rightarrow \infty$ and J_{v,w,τ_k} is a sequence that converges to a Jacobi field J , then $\|J(t)\| \leq \|J(0)\|$ for all $t \leq 0$, and hence J is the unstable Jacobi field with initial value w . Since we have the same limit for any such subsequence, it follows that $J_{v,w,\tau}$ converges as $\tau \rightarrow \infty$ to the unstable Jacobi field with initial value w . Thus $\mathcal{U}_{v,\tau}^u(0) \rightarrow \mathcal{U}_v^u(0)$ for each v .

Now given any $x \in \mathbb{R}^{n-1}$, we can use Dini's theorem to conclude that $\langle x, \mathcal{U}_{v,\tau}^u(0)x \rangle \rightarrow \langle x, \mathcal{U}_v^u(0)x \rangle$ uniformly in v . Since a symmetric matrix U is completely determined by $\langle x, Ux \rangle$ for a finite number of values of x , this shows that $\mathcal{U}_{v,\tau}^u(0) \rightarrow \mathcal{U}_v^u(0)$ uniformly in v . \square

Corollary 7.16. *For any $v \in T^1M$, $\mathcal{U}_v^u(0) \in D$.*

Proof. Lemma 7.9 tells us that $\mathcal{U}_{v,\tau}^u(0) \in D$ for all τ . Since D is compact, it follows from Lemma 7.15 that $\mathcal{U}_v^u(0) \in D$. \square

Proposition 7.17. *For each $\epsilon > 0$ there is $\tau_0(\epsilon) > 0$ such that if $\mathcal{U}(t)$ is a solution of the Riccati equation along the geodesic γ_v and $t_0 \in \mathbb{R}$ is such that $\mathcal{U}(t_0) \succcurlyeq 0$, then $\mathcal{U}(t) \succcurlyeq \mathcal{U}_v^u(t) - \epsilon \mathcal{I}$ for every $t \geq t_0 + \tau_0(\epsilon)$.*

Proof. Lemma 7.15 gives $\tau_0 = \tau_0(\epsilon)$ such that $\mathcal{U}_{w,\tau}^u(0) \succcurlyeq \mathcal{U}_w^u(0) - \epsilon \mathcal{I}$ for all $w \in T^1M$ and $\tau \geq \tau_0(\epsilon)$. Let $w = f_\tau v$ and $\tau = t$. \square

To prove Lemma 7.10, it suffices to consider the case when $s = 0$; to obtain the result when $s \neq 0$, replace $v \in T^1M$ by $f_s v$. By Proposition 7.17, there is $\tau_0 = \tau_0(\frac{\eta}{2})$ such that for any $v \in T^1M$ and $\mathcal{U}_0 \in D$, we have $\mathcal{R}_{0,t}^v \mathcal{U}_0 \succcurlyeq \mathcal{U}_v^u(t) - \frac{\eta}{2} \mathcal{I}$ for all $t \geq \tau_0$. We start by proving an estimate that is useful for controlling the pseudo-metric ρ locally. For $\mathcal{U} \in D$, we write $\mathcal{U}(t)$ to denote $\mathcal{R}_{0,t}^v \mathcal{U}$.

Lemma 7.18. *If $\underline{\mathcal{U}}, \overline{\mathcal{U}} \in D$ have $\underline{\mathcal{U}} \preccurlyeq \mathcal{U}_j \preccurlyeq \overline{\mathcal{U}}$ for $j = 0, 1$, then*

$$(7.15) \quad \rho(\mathcal{U}_0(t), \mathcal{U}_1(t)) \leq e^{\tau_0 \|\lambda\|} e^{-\int_0^t \bar{\lambda}(f_\tau v) d\tau} (\text{tr} \overline{\mathcal{U}} - \text{tr} \underline{\mathcal{U}}).$$

Proof. By Lemma 7.12, we have $\underline{\mathcal{U}}(t) \preccurlyeq \mathcal{U}_j(t) \preccurlyeq \overline{\mathcal{U}}(t)$ for all $t \in \mathbb{R}$ and $j = 0, 1$. Thus Weyl's inequality gives

$$\rho(\mathcal{U}_0(t), \mathcal{U}_1(t)) = |\text{tr} \mathcal{U}_0(t) - \text{tr} \mathcal{U}_1(t)| \leq \text{tr} \overline{\mathcal{U}}(t) - \text{tr} \underline{\mathcal{U}}(t) =: \Delta(t).$$

Writing $\underline{\lambda}_1(t) \leq \underline{\lambda}_2(t) \leq \dots \leq \underline{\lambda}_{n-1}(t)$ for the eigenvalues of $\underline{\mathcal{U}}(t)$, and similarly for the eigenvalues of $\overline{\mathcal{U}}(t)$, we have

$$\begin{aligned} \Delta'(t) &= \text{tr}(\overline{\mathcal{U}}'(t) - \underline{\mathcal{U}}'(t)) = \text{tr}(\underline{\mathcal{U}}(t)^2 - \overline{\mathcal{U}}(t)^2) \\ &= -(\text{tr}(\overline{\mathcal{U}}(t)^2) - \text{tr}(\underline{\mathcal{U}}(t)^2)) \\ &= -\sum_{i=1}^{n-1} (\bar{\lambda}_i(t)^2 - \underline{\lambda}_i(t)^2) = -\sum_{i=1}^{n-1} (\bar{\lambda}_i(t) - \underline{\lambda}_i(t))(\bar{\lambda}_i(t) + \underline{\lambda}_i(t)). \end{aligned}$$

Weyl's inequality gives $\bar{\lambda}_i(t) \geq \underline{\lambda}_i(t) \geq 0$, so $\Delta'(t) \leq 0$ for all $t \geq 0$. Moreover, since $\underline{\mathcal{U}}(0) \in D$, Proposition 7.17 gives $\underline{\lambda}_i(t) \geq \lambda(f_t v) - \frac{\eta}{2}$ for all $t \geq \tau_0$, and thus $\underline{\lambda}_i(t) \geq \tilde{\lambda}(f_t v)$. Thus, for $t \geq \tau_0$, we have

$$\Delta'(t) \leq - \sum_{i=1}^{n-1} 2\tilde{\lambda}(f_t v)(\bar{\lambda}_i(t) - \underline{\lambda}_i(t)) \leq -2\tilde{\lambda}(f_t v)\Delta(t),$$

and so

$$\begin{aligned} \rho(\mathcal{U}_0(t), \mathcal{U}_1(t)) &\leq \Delta(\tau_0) e^{-\int_{\tau_0}^t 2\tilde{\lambda}(f_\tau v) d\tau} \\ &\leq \Delta(0) e^{\int_0^{\tau_0} \tilde{\lambda}(f_\tau v) d\tau} e^{-\int_0^t \tilde{\lambda}(f_\tau v) d\tau} \\ &\leq (\text{tr } \bar{\mathcal{U}} - \text{tr } \underline{\mathcal{U}}) e^{\tau_0 \|\lambda\|} e^{-\int_0^t \tilde{\lambda}(f_\tau v) d\tau}. \quad \square \end{aligned}$$

We now apply the estimate (7.15) locally on the interior of D , and show how to use this to obtain the global estimate (7.10). First assume that $\mathcal{U}_0, \mathcal{U}_1$ are positive definite, and let $\epsilon > 0$ be such that $\mathcal{U}_0, \mathcal{U}_1 \succ \epsilon \mathcal{I}$ and $n = \|\mathcal{U}_0 - \mathcal{U}_1\|/\epsilon$ is an integer. Given $q \in (0, 1)$, let $\mathcal{U}_q = (1 - q)\mathcal{U}_0 + q\mathcal{U}_1$ and observe that $\mathcal{U}_q \succ \epsilon \mathcal{I}$. For every $0 \leq k < n$, we have $\|\mathcal{U}_{(k+1)/n} - \mathcal{U}_{k/n}\| < \epsilon$.

Now let $\underline{\mathcal{U}}_k = \mathcal{U}_{k/n} - \epsilon \mathcal{I}$ and $\bar{\mathcal{U}}_k = \mathcal{U}_{k/n} + \epsilon \mathcal{I}$. For $j = k/n, (k+1)/n$, we have $\underline{\mathcal{U}}_k \preccurlyeq \mathcal{U}_j \preccurlyeq \bar{\mathcal{U}}_k$, so writing $\mathcal{U}_q(t) = \mathcal{R}_{0,t}^v(\mathcal{U}_q)$ for $q \in [0, 1]$, and applying Lemma 7.18 gives

$$\rho(\mathcal{U}_{k/n}(t), \mathcal{U}_{(k+1)/n}(t)) \leq e^{\tau_0 \|\lambda\|} e^{-\int_0^t \tilde{\lambda}(f_\tau v) d\tau} 2\epsilon.$$

Summing over all k , and using the fact that $n\epsilon = \|\mathcal{U}_0 - \mathcal{U}_1\|$, gives

$$\rho(\mathcal{U}_0, \mathcal{U}_1) \leq 2e^{\tau_0 \|\lambda\|} e^{-\int_0^t \tilde{\lambda}(f_\tau v) d\tau} \|\mathcal{U}_0 - \mathcal{U}_1\|.$$

This proves (7.10) when $\mathcal{U}_0, \mathcal{U}_1$ are positive definite. For the positive semidefinite case, replace \mathcal{U}_j with $\mathcal{U}_j^\delta := \mathcal{U}_j + \delta \mathcal{I}$ for $\delta > 0$, and observe that $\lim_{\delta \rightarrow 0} \mathcal{R}_{0,t}^v \mathcal{U}_j^\delta = \mathcal{R}_{0,t}^v \mathcal{U}_j$. This completes the proof of Lemma 7.10.

7.5. Proof of Lemma 7.11. Fixing $v, w \in T^1 M$ and $t \geq 0$, let $R: [0, t] \rightarrow D$ be as in (7.11), and define $G: [0, t] \rightarrow [0, \infty)$ by

$$(7.16) \quad G(s) = Q e^{-\int_s^t \tilde{\lambda}(f_\tau v) d\tau} \|\mathcal{K}(f_s v) - \mathcal{K}(f_s w)\|.$$

We start by showing that $\rho(R(s_1), R(s_2))$ can be controlled in terms of $G(s_1)$ when $s_2 - s_1$ is small. First we need a uniform continuity property of the map $(v, s, \mathcal{U}) \mapsto \|\mathcal{R}_{0,s}^v \mathcal{U}\|$.

Lemma 7.19. *For every $\epsilon > 0$ there is $\delta > 0$ such that given any $v, w \in T^1 M$, $\mathcal{U} \in D$, and $s \in [0, \delta]$, we have $\|\mathcal{R}_{0,s}^v \mathcal{U} - \mathcal{R}_{0,s}^w \mathcal{U}\| \leq (\|\mathcal{K}(v) - \mathcal{K}(w)\| + \epsilon)s$.*

Proof. Given $v \in T^1M$ and $\mathcal{U} \in D$, let $\mathcal{U}_v(s) = \mathcal{R}_{0,s}^v \mathcal{U}$. Then for every $v, w \in T^1M$ and every $s \in [0, \delta_1)$, we have

$$\|\mathcal{U}'_v(s) - \mathcal{U}'_w(s)\| \leq \|\mathcal{U}_v^2(s) - \mathcal{U}_w^2(s)\| + \|\mathcal{K}(f_s v) - \mathcal{K}(f_s w)\|.$$

Since $\|\mathcal{U}\| \leq b$ for all $\mathcal{U} \in D$, we get $\|(\mathcal{U}_v - \mathcal{U}_w)'(s)\| \leq 4b^2 + 2\|\mathcal{K}\| =: L$, so $\|\mathcal{U}_v^2(s) - \mathcal{U}_w^2(s)\| \leq 2b\|\mathcal{U}_v(s) - \mathcal{U}_w(s)\| \leq 2bLs$, and

$$(7.17) \quad \|(\mathcal{U}_v - \mathcal{U}_w)'(s)\| \leq 2bLs + \|\mathcal{K}(f_s v) - \mathcal{K}(f_s w)\|.$$

Fix $\epsilon > 0$. Uniform continuity of \mathcal{K} gives $\epsilon' > 0$ such that $d_K(v_1, v_2) < \epsilon'$ implies $\|\mathcal{K}(v_1) - \mathcal{K}(v_2)\| < \frac{\epsilon}{4}$. Uniform continuity of the flow gives $\delta_1 > 0$ such that $\|\mathcal{K}(f_s v) - \mathcal{K}(v)\| < \frac{\epsilon}{4}$ for every $v \in T^1M$ and $|s| < \delta_1$. Let $\delta = \min(\frac{\epsilon}{4bL}, \delta_1)$. For any $0 \leq s \in [0, \delta)$, (7.17) gives $\|(\mathcal{U}_v - \mathcal{U}_w)'(s)\| \leq \|\mathcal{K}(v) - \mathcal{K}(w)\| + \epsilon$, and integrating proves the lemma. \square

Now fix $\epsilon > 0$, and let Q be the constant from Lemma 7.10. By Lemma 7.19, there is $\delta > 0$ such that for every $v, w \in T^1M$, $\mathcal{U}_0 \in D$, and $0 \leq s_0 \leq s \leq s_0 + \delta$, noting that $\mathcal{R}_{0,s}^w \mathcal{U}_0 = \mathcal{R}_{s_0,s}^w \mathcal{R}_{0,s_0}^w \mathcal{U}_0$, we have

$$(7.18) \quad \|\mathcal{R}_{s_0,s}^v \mathcal{R}_{0,s_0}^w \mathcal{U}_0 - \mathcal{R}_{0,s}^w \mathcal{U}_0\| \leq (\|\mathcal{K}(f_{s_0} v) - \mathcal{K}(f_{s_0} w)\| + \epsilon/Q)(s - s_0).$$

If $s \leq t$, then letting $R = R_{\mathcal{U}_0}^{v,w,t}$, we have

$$\begin{aligned} \rho(R(s_0), R(s)) &= \rho(\mathcal{R}_{s_0,t}^v \mathcal{R}_{0,s_0}^w \mathcal{U}_0, \mathcal{R}_{s,t}^v \mathcal{R}_{0,s}^w \mathcal{U}_0) \\ &= \rho(\mathcal{R}_{s,t}^v (\mathcal{R}_{s_0,s}^v \mathcal{R}_{0,s_0}^w \mathcal{U}_0), \mathcal{R}_{s,t}^v (\mathcal{R}_{0,s}^w \mathcal{U}_0)), \end{aligned}$$

and so applying Lemma 7.10 and the estimate (7.18) gives

$$\begin{aligned} \rho(R(s_0), R(s)) &\leq Qe^{-\int_{s_0}^s \tilde{\lambda}(f_\tau v) d\tau} \|\mathcal{R}_{s_0,s}^v \mathcal{R}_{0,s_0}^w \mathcal{U}_0 - \mathcal{R}_{0,s}^w \mathcal{U}_0\| \\ &\leq (G(s_0) + \epsilon)(s - s_0), \end{aligned}$$

where G is as in (7.16). Now given any $0 \leq s_1 \leq s_2 \leq t$ as in the statement of Lemma 7.11, let n be large enough that $(s_2 - s_1)/n \leq \delta$, and put $s_i^* = s_1 + \frac{i}{n}(s_2 - s_1)$ for $0 \leq i \leq n$; we obtain

$$\rho(R(s_1), R(s_2)) \leq \sum_{i=1}^n \rho(s_{i-1}^*, s_i^*) \leq \sum_{i=1}^n (G(s_{i-1}^*) + \epsilon) \left(\frac{s_2 - s_1}{n} \right).$$

Sending $n \rightarrow \infty$, the Riemann sum converges to $\int_{s_1}^{s_2} (G(s) + \epsilon) ds$ since G is continuous. Since $\epsilon > 0$ was arbitrary, this proves (7.12).

8. PRESSURE GAP

In this section we prove Theorem B. The main idea is to approximate orbit segments in the singular set with ones in $\mathcal{C}(\eta)$; that is, orbits that start and end in $\text{Reg}(\eta) \subset \text{Reg}$. We then use the specification property of $\mathcal{C}(\eta)$ to generate a collection of orbits with greater topological pressure than the singular set.

8.1. Replacing singular orbit segments with regular ones. If η_0 is small and R is large, the following shows that the stable and unstable leaves of radius R through any v must both intersect $\text{Reg}(\eta_0)$.

Proposition 8.1. *There exist $R, \eta_0 > 0$ such that for every $v \in T^1M$, there exist $w^s \in W_R^s(v)$ and $w^u \in W_R^u(v)$ satisfying $\lambda(w^s), \lambda(w^u) \geq \eta_0$.*

Proof. Let $\text{Reg}' := \{v \in T^1M \mid \lambda(v) > 0\}$, and note that $\text{Reg}' \subset \text{Reg}$. Since Reg' is open and non-empty, and $W^{s,u}$ are uniformly dense, there exists $R > 0$ such that $W_R^s(v) \cap \text{Reg}' \neq \emptyset$ and $W_R^u(v) \cap \text{Reg}' \neq \emptyset$ for every $v \in T^1M$. Fix such an R , and define $\bar{\eta}: T^1M \rightarrow (0, \infty)$ by

$$\bar{\eta}(v) = \min \left(\max\{\lambda(w) : w \in W_R^s(v)\}, \max\{\lambda(w) : w \in W_R^u(v)\} \right).$$

Then $\bar{\eta}$ depends continuously on v and is always positive. Since T^1M is compact, there is $\eta_0 > 0$ such that $\bar{\eta}(v) > \eta_0$ for every $v \in T^1M$. \square

By Proposition 8.1, we can define maps $\Pi^s, \Pi^u: T^1M \rightarrow \text{Reg}(\eta_0)$ such that $\Pi^\sigma(v) \in W_R^\sigma(v)$ for every $v \in T^1M$ and $\sigma = s, u$. Given $t > 0$, we use these to define a map $\Pi_t: \text{Sing} \rightarrow \text{Reg}$ by

$$(8.1) \quad \Pi_t = f_{-t} \circ \Pi^u \circ f_t \circ \Pi^s.$$

That is, given $v \in \text{Sing}$ we choose $v' = \Pi^s(v) \in W_R^s(v)$ with $\lambda(v') \geq \eta_0$, and $w = f_{-t}(\Pi^u(f_t v'))$ such that $f_t w \in W_R^u(f_t v')$ and $\lambda(f_t w) \geq \eta_0$. We can also think of this procedure as acting on orbit segments by writing $\Pi((v, t)) = (\Pi_t(v), t)$; the next result shows that for $\eta < \eta_0$, this gives a map $\Pi: \text{Sing} \times [T, \infty) \rightarrow \mathcal{C}(\eta)$ when T is large enough. When we prove the pressure gap below, we will focus on orbit segments with integer lengths, and so we will consider the maps Π_n for $n \in \mathbb{N}$.

Theorem 8.2. *Let R, η_0, Π_n be as above. Then for every $\delta > 0$ and $\eta \in (0, \eta_0)$, there is $T > 0$ such that for every $v \in \text{Sing}$ and $n > 2T$, the image $w = \Pi_n(v)$ has the following properties:*

- (1) $w, f_n(w) \in \text{Reg}(\eta)$;
- (2) $d_K(f_t(w), \text{Sing}) < \delta$ for all $t \in [T, n - T]$;
- (3) for every $t \in [T, n - T]$, $f_t(w)$ and v lie in the same connected component of $B(\text{Sing}, \delta) := \{w \in T^1M : d_K(w, \text{Sing}) < \delta\}$.

Observe that Theorem 8.2 does *not* allow us to conclude that $f_t(w)$ is close to $f_t(v)$; all we know is that $f_t(w)$ is close to *some* singular vector for $t \in [T, n - T]$. For example, if $f_t(v)$ is in the middle of a flat strip, then $f_t(w)$ will be close to the edge of the flat strip for $t \in [T, n - T]$.

Proof of Theorem 8.2. Let δ, η, η_0 be as in the statement of the theorem. For property (1), it is immediate from the definition of Π_n that $\lambda(f_n w) \geq \eta$. Applying Corollary 3.14 with $\eta_0 > \eta > 0$, we obtain T_0

such that $\lambda(v') \geq \eta_0$ and $f_{T_0}(w) \in W_R^u(f_{T_0}v')$ imply that $\lambda(w) \geq \eta$. Thus, item (1) of the theorem holds for any $n \geq T_0$.

We turn our attention to item (2). By Proposition 3.5, there are $\eta_1, T_1 > 0$ such that

$$(8.2) \quad \text{if } \lambda^u(f_tv) \leq \eta_1 \text{ for all } |t| \leq T_1, \text{ then } d_K(v, \text{Sing}) < \delta.$$

We apply Corollary 3.14 twice to obtain $T_2 > 0$ such that if $v, w \in T^1M$ have $f_{-t}(v) \in W_R^s(f_{-t}w)$ or $f_t(v) \in W_R^u(f_tw)$ for some $t \geq T_2$, then the following two properties hold:

$$(8.3) \quad \lambda^u(v) < \frac{\eta_1}{4} \Rightarrow \lambda^u(w) < \frac{\eta_1}{2},$$

$$(8.4) \quad \lambda^u(v) < \frac{\eta_1}{2} \Rightarrow \lambda^u(w) < \eta_1.$$

Given $v \in \text{Sing}$, we have $\Pi^s(v) = v' \in W_R^s(v)$, and $\lambda(f_tv) = 0$ for all t . For $t \geq T_2$ we have $f_tv \in f_t(W_R^s(v'))$, so (8.3) gives $\lambda^u(f_tv') < \eta_1/2$.

Now $w = \Pi_n(v) = f_{-n}\Pi^u(f_nv')$ satisfies $f_nw \in W_R^u(f_nv')$, so (8.4) gives $\lambda^u(f_tw) \leq \eta_1$ for all $t \in [T_2, n - T_2]$. Applying (8.2) gives $d_K(f_tw, \text{Sing}) < \delta$ for all $t \in [T_2 + T_1, n - T_2 - T_1]$. Thus, taking $T = \max(T_0, T_2 + T_1)$, assertions (1) and (2) follow for $n > 2T$.

For item (3) of the theorem, we observe that v and w can be connected by a path $u(s)$ that follows first $W_R^s(v)$, then $f_{-n}(W_R^u(f_nv'))$, and that the arguments giving $d_K(f_tw, \text{Sing}) < \delta$ also give $d_K(f_tu(s), \text{Sing}) < \delta$ for every $t \in [T, n - T]$ and every s . We conclude that f_tv and f_tw lie in the same connected component of $B(\text{Sing}, \delta)$ for every such t . \square

8.1.1. Controlling the multiplicity of the map Π_n . Recall that $d_n(v, w) = \max\{d_K(f_tv, f_tw) : t \in [0, n]\} = \max\{d(\gamma_v(t), \gamma_w(t)) : t \in [0, n+1]\}$. We bound the number of $v \in \text{Sing}$ whose images under Π_n are close in the d_n metric.

Proposition 8.3. *For every $\epsilon > 0$, there exists $C = C(M, R, \epsilon) > 0$ such that if $E_n \subset \text{Sing}$ is an $(n, 2\epsilon)$ -separated set for some $n > 0$, then for every $w \in T^1M$, we have $\#\{v \in E_n \mid d_n(w, \Pi_nv) < \epsilon\} \leq C$.*

Proof. Let \widetilde{M} be the universal cover of M and $B \subset \widetilde{M}$ a fundamental domain. Define $\widetilde{\Pi}^{s,u}$ and $\widetilde{\Pi}_n$ in the obvious way, by lifting $\Pi^{s,u}$ and Π_n to the universal cover. We write \widetilde{d} for the lift of the Riemannian metric d to \widetilde{M} , and \widetilde{d}_n for the lift of the metric d_n . Every $\widetilde{v} \in T^1\widetilde{M}$ has

$$\begin{aligned} \widetilde{d}(\pi\widetilde{v}, \pi\widetilde{\Pi}_n\widetilde{v}) &\leq \widetilde{d}(\pi\widetilde{v}, \pi\widetilde{\Pi}^s\widetilde{v}) + \widetilde{d}(\pi\widetilde{\Pi}^s\widetilde{v}, \pi\widetilde{\Pi}_n\widetilde{v}) \\ &\leq d^s(v, \Pi^sv) + d^u(\Pi^sv, \Pi_nv) \leq 2R, \end{aligned}$$

recalling that $\pi\tilde{v} \in \tilde{M}$ is the footprint of \tilde{v} . Given $v \in T^1M$, let $\tilde{v}_B \in T^1\tilde{M}$ be the lift of v with $\pi\tilde{v}_B \in B$; then we have $\pi\tilde{\Pi}_n\tilde{v}_B \in A_{2R} := \bigcup_{x \in B} B_{\tilde{d}}(x, 2R)$.

Fix $\epsilon > 0$ and let $\Lambda = \Lambda_{(2R+\epsilon), B} := \{g \in \pi_1(M) \mid gB \cap A_{2R+\epsilon} \neq \emptyset\}$. Note that $\#\Lambda < \infty$ because \bar{B} is compact. For $n > 0$, let $E_n \subset \text{Sing}$ be any $(n, 2\epsilon)$ -separated set, and fix an arbitrary $w \in T^1M$. We define

$$E_n^{w, \epsilon} := \{v \in E_n \mid d_n(w, \Pi_n v) < \epsilon\}.$$

Let $X \subset B$ and $Y \subset A_{2R+\epsilon}$ be finite ϵ -dense sets. We will show that $\#E_n^{w, \epsilon} \leq (\#\Lambda)(\#X)(\#Y) =: C$.

Since $d_n(w, \Pi_n v) < \epsilon$, there exists a lift \tilde{w} of w with $\tilde{d}_n(\tilde{w}, \tilde{\Pi}_n\tilde{v}_B) < \epsilon$. It follows that $\pi\tilde{w} \in A_{2R+\epsilon}$, and thus $\pi\tilde{w} \in gB$ for some $g \in \Lambda$. Thus, $E_n = \bigcup_{g \in \Lambda} E_n^g$, where

$$E_n^g = \{v \in E_n^{w, \epsilon} \mid \tilde{d}_n(\tilde{w}, \tilde{\Pi}_n\tilde{v}) < \epsilon \text{ where } \tilde{w} \text{ is the lift of } w \text{ to } gB\}.$$

For a fixed $g \in \Lambda$ and $v \in E_n^g$, we approximate \tilde{v}_B and $f_n\tilde{v}_B$ using the sets X and Y . Recall that $\pi\tilde{v}_B \in B$ by definition, and we will show that the location of $f_n\tilde{v}_B$ in $T^1\tilde{M}$ is controlled by using $f_n\tilde{w}$ as a reference point. Given $v \in E_n^g$, let $x = x(v) \in X$ be such that $\tilde{d}(x, \pi\tilde{v}_B) < \epsilon$. Let h be the unique element of $\pi_1(M)$ so that $\pi f_n\tilde{w} \in hB$. Then $\tilde{d}(hB, \pi f_n(\tilde{\Pi}_n\tilde{v}_B)) < \epsilon$, and thus $\pi f_n\tilde{v}_B \in h(A_{2R+\epsilon})$. Thus, there is a unique $y = y(v) \in Y$ such that $\tilde{d}(\pi f_n\tilde{v}_B, h(y)) < \epsilon$.

Now we show that the map $x \times y: E_n^g \rightarrow X \times Y$ is injective. Given $v_1, v_2 \in E_n^g$ and $t \in [0, n]$, let $\rho(t) = \tilde{d}(\gamma_{\tilde{v}_1}(t), \gamma_{\tilde{v}_2}(t))$ and note that ρ is convex. In particular, it takes its maximum value at an endpoint. If $x(v_1) = x(v_2)$, then $\rho(0) < 2\epsilon$, and if $y(v_1) = y(v_2)$, then $\rho(n) < 2\epsilon$. Thus if v_1, v_2 have the same image under $x \times y$, we get $\tilde{d}_n((\tilde{v}_1)_B, (\tilde{v}_2)_B) < 2\epsilon$, and thus $d_n(v_1, v_2) < 2\epsilon$. Since E_n^g is $(n, 2\epsilon)$ -separated, this gives $v_1 = v_2$. Injectivity shows that $\#E_n^g \leq (\#X)(\#Y)$ for every $g \in \Lambda$, which proves Proposition 8.3 with $C = \#\Lambda\#X\#Y$. \square

8.2. Creating entropy. First, we obtain a lower bound on certain partition sums for the singular set. By [16, Proposition 3.3], the flow is entropy-expansive and thus there exists $\epsilon > 0$ such that $P(\text{Sing}, \varphi) = P(\text{Sing}, \varphi, \epsilon)$. We consider the following partition sum:

$$\tilde{\Lambda}(X, \epsilon, t, \varphi) := \sup \left\{ \sum_{x \in E} e^{\sup_{y \in B_t(x, \epsilon)} \Phi(y, t)} \mid E \subset X \text{ is } (t, \epsilon)\text{-separated} \right\}.$$

The argument in [6, Lemmas 4.1 and 4.2] proves that

$$(8.5) \quad \tilde{\Lambda}(\text{Sing}, \epsilon, n, \varphi) \geq e^{nP(\text{Sing}, \varphi)}.$$

Fix $\eta_0 > 0$ as in Proposition 8.1, and choose $\delta > 0$ small enough that:

- $\tilde{\Lambda}(\text{Sing}, 2\delta, n, \varphi) \geq e^{nP(\text{Sing}, \varphi)}$ for every n ;
- φ is locally constant on $B(\text{Sing}, 2\delta)$;
- $\lambda(v) < \eta_0/2$ for all $v \in B(\text{Sing}, 2\delta)$.

Let U_1, \dots, U_k be the components of $B(\text{Sing}, 2\delta)$, and let $\Phi_i \in \mathbb{R}$ be the (constant) value that φ takes on U_i . By (8.5), for every n , there exists an $(n, 2\delta)$ -separated set $E_n \subset \text{Sing}$ such that

$$(8.6) \quad \sum_{i=1}^k e^{n\Phi_i} \#(E_n \cap U_i) \geq e^{nP(\text{Sing}, \varphi)}.$$

We consider the image of E_n under the map Π_n . Fix $\eta \in (\eta_0/2, \eta_0)$, and let $T = T(\eta, \delta)$ be as in Theorem 8.2. Write $E'_n = \Pi_n(E_n)$; then

$$(8.7) \quad w, f_n(w) \in \text{Reg}(\eta) \text{ for every } w \in E'_n \text{ with } n > 2T.$$

By Theorem 4.1, $\{(w, n) : w \in E'_n\}$ has the specification property.

Given $v \in E_n \cap U_i$, the third item of Theorem 8.2 shows that for any $u \in B_n(\Pi_n v, \delta)$, we have

$$(8.8) \quad \int_0^n \varphi(f_t u) dt \geq (n - 2T)\Phi_i - 2T\|\varphi\| \geq n\Phi_i - 4T\|\varphi\|.$$

By Proposition 8.3, for each $w \in E'_n$ there are at most $C = C(M, R, \delta)$ elements $w' \in E'_n$ with $d_n(w, w') < \delta$; this leads to the following lemma.

Lemma 8.4. *For every n , there is an (n, δ) -separated set $E''_n \subset E'_n$ such that, setting $\beta = C^{-1}e^{-4T\|\varphi\|}$, we have*

$$(8.9) \quad \sum_{w \in E''_n} e^{\inf_{u \in B_n(w, \delta)} \int_0^n \varphi(f_t u) dt} \geq \beta e^{nP(\text{Sing}, \varphi)}.$$

Proof. Given $1 \leq i \leq k$, let $E'_{n,i} = \Pi_n(E_n \cap U_i)$ and take a maximal (n, δ) -separated subset $E''_{n,i} \subset E'_{n,i}$. Now $\#E'_{n,i} \cap B_n(w, \delta) \leq C$ for all $w \in E'_{n,i}$ and $E''_{n,i}$ is (n, δ) -spanning for $E'_{n,i}$, so $\#E''_{n,i} \geq C^{-1}\#E'_{n,i}$. Sum over i and use (8.6) and (8.8) to get (8.9) for $E''_n = \bigcup_{i=1}^k E''_{n,i}$. \square

The proof of Theorem B will be completed by using the partition sum estimate (8.9) together with the specification property to produce more topological pressure as follows.

- Fix $\alpha > 0$, and then for each N , split $[0, N]$ into αN pieces of lengths $n_1, \dots, n_{\alpha N}$.
- Fill in each of the pieces with a trajectory originating in E''_{n_j} .
- Use the specification property for $\mathcal{C}(\eta)$ to find a single trajectory (u, N) that shadows this sequence of trajectories.

- This procedure gives $\lambda(f_t u) > \eta$ if and only if t is close to a surgery time; we use this to show that distinct choices of length data $n_1, \dots, n_{\alpha N}$ yield distinct orbit segments (u, N) no matter which elements of E''_{n_j} were chosen above.
- Using the fact that $\binom{N}{\alpha N} \approx e^{(-\alpha \log \alpha)N}$, conclude that $T^1 M$ has an $(N, \delta/5)$ -separated set F_N with (roughly)

$$\sum_{w \in F_N} e^{\int_0^N \varphi(f_t w) dt} \geq e^{-(\alpha \log \alpha)N} e^{-Q\alpha N} e^{NP(\text{Sing}, \varphi)},$$

where the constant Q is independent of α . For small α , this gives $P(\varphi) \geq \alpha(-\log \alpha - Q) + P(\text{Sing}, \varphi) > P(\text{Sing}, \varphi)$.

In the final estimate, we need α to be small because the transition times in the specification property give αN time intervals where (u, N) is not controlled. The error introduced in this way is the price we pay for creating new orbits. This is why α must be chosen small; exponential growth comes from the $\binom{N}{\alpha N} \approx e^{(-\alpha \log \alpha)N}$ term. The details of this scheme are carried out in the next section.

8.2.1. Details of entropy production scheme. Recall that $\eta, \delta > 0$ are chosen such that

$$(8.10) \quad d(v, \text{Sing}) < 2\delta \Rightarrow \lambda(v) < \eta.$$

By Proposition 4.6, there is τ such that the following specification property holds on $\mathcal{C}(\eta) = \{(v, t) \in T^1 M \times (0, \infty) \mid v, f_t v \in \text{Reg}(\eta)\}$: for every $\{(v_j, t_j)\}_{j=1}^k \subset \mathcal{C}(\eta)$ and every s_1, s_2, \dots, s_k with the property that $s_{j+1} \geq s_j + t_j + 2\tau$, there are $\tau_j \in [0, \tau]$ and $w \in T^1 M$ such that

$$(8.11) \quad f_{s_j + \tau_j}(w) \in B_{t_j}(v_j, \delta/5) \text{ for all } 1 \leq j \leq k.$$

Without loss of generality we assume that $T = T(\eta, \delta)$ from Theorem 8.2 satisfies $T \geq 2\tau$.

Let $\alpha > 0$ be rational and let $N \in \mathbb{N}$ be large, with the property that $\alpha N \in \mathbb{N}$. Consider the set

$$\mathcal{A} = \{4T, 8T, 12T, \dots, 4(N-1)T\} \subset [0, 4NT],$$

which we designate as the collection of “times which could be marked for performing surgery”. We will select $\alpha N - 1$ of the $N - 1$ elements in \mathcal{A} as times where an orbit segment $(x, 4NT)$ is marked for surgery. We write $\mathbb{J}_N^\alpha = \{J \subset \mathcal{A} \mid \#J = \alpha N - 1\}$ for this collection of possible marker placements, and note that $\#\mathbb{J}_N^\alpha = \binom{N-1}{\alpha N - 1}$.

We now obtain estimates for a fixed placement of these markers. Given $J \in \mathbb{J}_N^\alpha$, let $n_1, \dots, n_{\alpha N} \in \mathbb{N}$ be such that the gaps between successive elements of J are $4n_i T$, so that writing $N_j = \sum_{i=1}^j n_i$, we have

$J = \{4N_1T, \dots, 4N_{\alpha N-1}T\}$, and $N_{\alpha N} = N$. Using the specification property for orbits in $\mathcal{C}(\eta)$, together with the partition sum estimate (8.9), we will now prove the following.

Proposition 8.5. *There is a constant Q , depending only on T , τ , β , and the modulus of continuity for the flow f_t , such that the following is true. For every $N, \alpha > 0$ and every $J \in \mathbb{J}_N^\alpha$, there is a $(4NT, \delta/5)$ -separated set \mathcal{X}_J with the property that*

$$(8.12) \quad \sum_{w \in \mathcal{X}_J} e^{\int_0^{4NT} \varphi(f_t w) dt} \geq e^{-\alpha N Q} e^{4NT P(\text{Sing}, \varphi)},$$

and moreover, every $w \in \mathcal{X}_J$ satisfies

$$(8.13) \quad \begin{aligned} & d(f_t w, \text{Sing}) > 9\delta/5 \text{ for some } t \in [s, s+T] \text{ if } s \in J, \\ & d(f_t w, \text{Sing}) < 6\delta/5 \text{ for every } t \in [s, s+T] \text{ if } s \in \mathcal{A} \setminus J. \end{aligned}$$

Proof. We apply the specification property to elements of $E''_{(4n_j-1)T}$; we use (8.11) with $s_j = 4N_{j-1}T$ and $t_j = (4n_j - 1)T$, so that

$$s_j + t_j + 2\tau \leq 4N_{j-1}T + (4n_j - 1)T + T = 4N_jT = s_{j+1},$$

and conclude that for every choice of $v_j \in E''_{(4n_j-1)T}$, there are $\tau_j \in [0, T]$ and $w \in T^1M$ such that

$$(8.14) \quad f_{4N_{j-1}T+\tau_j}(w) \in B_{(4n_j-1)T}(v_j, \delta/5) \text{ for all } 1 \leq j \leq \alpha N.$$

We prove that each such w satisfies (8.13). If $s \in J$, then we have $s = 4N_jT$ for some j , and writing $t = s + t_j$, (8.14) gives $d(f_t(w), v_j) < \delta/5$; since $v_j \in \text{Reg}(\eta)$, this gives $d(f_t w, \text{Reg}(\eta)) < \delta/5$; by (8.10), this is enough. If $s \in \mathcal{A} \setminus J$, then we have $s = 4nT$ for some $n \in \{1, \dots, N-1\}$, and moreover there is j such that $N_j < n < N_{j+1}$. Again we use (8.14) to conclude that for every $t \in [s, s+T]$, we can write $t' = t - (4N_jT + t_j)$ and get $d(f_t(w), f_{t'}(v_j)) < \delta/5$; since

$$\begin{aligned} t' &\geq 4(n - N_j)T \geq T, \\ t' &\leq 4(n - N_j)T + T \leq 4(n_j - 1)T + T \leq (4n_j - 1)T - T, \end{aligned}$$

our construction of v_j using Theorem 8.2 gives $d(f_{t'}(v_j), \text{Sing}) < \delta$, which establishes the other half of (8.13).

It remains to produce a $(4NT, \delta/5)$ -separated set satisfying the partition sum estimate (8.12). Write $G: \prod_{j=1}^{\alpha N} E''_{(4n_j-1)T} \rightarrow T^1M \times [0, T]^{\alpha N}$ for the map that takes $(v_1, \dots, v_{\alpha N})$ to $(w, \tau_1, \dots, \tau_{\alpha N})$. We would like to say that the set of all w produced in this way is $(4NT, \delta/5)$ -separated, but because τ_j can vary for different choices of v_i , we cannot immediately conclude this. Rather, we use uniform continuity of the flow to find $K \in \mathbb{N}$ such that $d(v, f_t v) < \delta/5$ whenever $|t| < T/K =: \epsilon$, and

then consider the map $(0, T] \rightarrow \{1, 2, 3, \dots, K\}$ given by $x \mapsto \lceil x/\epsilon \rceil$, so every $x \in ((k-1)\epsilon, k\epsilon]$ is taken to the integer k . Composing this with the map G gives

$$H: \prod_{j=1}^{\alpha N} E''_{(4n_j-1)T} \rightarrow T^1M \times \{1, 2, \dots, K\}^{\alpha N},$$

$$(v_1, \dots, v_{\alpha N}) \mapsto (w, k_1, \dots, k_{\alpha N}),$$

with the property that by (8.14) and the choice of K , we have

$$(8.15) \quad f_{4N_{j-1}T+k_j\epsilon}(w) \in B_{(4n_j-1)T}(v_j, 2\delta/5) \text{ for all } 1 \leq j \leq \alpha N.$$

Given $\mathbf{k} \in \{1, \dots, K\}^{\alpha N}$, let $\mathcal{X}_j^{\mathbf{k}}$ be the set of all $w \in T^1M$ such that (w, \mathbf{k}) is in the image of the map H .

Lemma 8.6. $\mathcal{X}_j^{\mathbf{k}}$ is $(4NT, \delta/5)$ -separated.

Proof. Given $w^1 \neq w^2 \in \mathcal{X}_j^{\mathbf{k}}$, there are $\mathbf{v}^1 \neq \mathbf{v}^2 \in \prod_{j=1}^{\alpha N} E''_{(4n_j-1)T}$ such that $H(\mathbf{v}^i) = (w^i, \mathbf{k})$. Each of the sets $E''_{(4n_j-1)T}$ is $((4n_j-1)T, \delta)$ -separated, so there are $j \in \{1, \dots, \alpha N\}$ and $t \in [0, (4n_j-1)T]$ such that $d(f_tv_j^1, f_tv_j^2) \geq \delta$. Together with (8.15), this gives

$$d(f_{4N_{j-1}T+k_j\epsilon+t_j}(w^1), f_{4N_{j-1}T+k_j\epsilon+t_j}(w^2))$$

$$\geq d(f_{t_j}(v_j^1), f_{t_j}(v_j^2)) - \sum_{i=1}^2 d(f_{4N_{j-1}T+k_j\epsilon+t_j}(w^i), f_{t_j}(v_j^i)) \geq \delta/5,$$

which proves the lemma. \square

We estimate $\int_0^{4NT} \varphi(f_tw) dt$ by breaking the integral over $[0, 4NT]$ into pieces corresponding to the intervals $[4N_{j-1}T + k_j\epsilon, (4N_j-1)T + k_j\epsilon]$, on which the orbit of u is within $2\delta/5$ of the orbit of v_j ; the leftover pieces have a total length of $\leq \alpha NT$, and so for $(w, \mathbf{k}) = H(\mathbf{v})$, we get

$$\int_0^{4NT} \varphi(f_tw) dt \geq -\alpha NT \|\varphi\| + \sum_{j=1}^{\alpha N} \inf_{w \in B_{t_j}(v_j, 2\delta/5)} \int_0^{t_j} \varphi(f_tv_j) dt,$$

where we write $t_j = (4n_j-1)T$. Using this together with the partition sum estimate (8.9) gives

$$\sum_{\mathbf{k}} \sum_{w \in \mathcal{X}_j^{\mathbf{k}}} e^{\int_0^{4NT} \varphi(f_tw) dt} \geq \sum_{\mathbf{v}} e^{-\alpha NT \|\varphi\|} \prod_{j=1}^{\alpha N} e^{\inf_{w \in B_{t_j}(v_j, 2\delta/5)} \int_0^{t_j} \varphi(f_tv_j) dt}$$

$$\geq e^{-\alpha NT \|\varphi\|} \prod_{j=1}^{\alpha N} \beta e^{t_j P(\text{Sing}, \varphi)} \geq e^{\alpha N(\log \beta - 2T \|\varphi\|)} e^{4NT P(\text{Sing}, \varphi)}.$$

Finally, since there are $K^{\alpha N}$ choices for \mathbf{k} , we conclude that there is \mathbf{k} such that $\mathcal{X}_J := \mathcal{X}_J^{\mathbf{k}}$ has

$$(8.16) \quad \sum_{w \in \mathcal{X}_J} e^{\int_0^{4NT} \varphi(f_t w) dt} \geq K^{-\alpha N} e^{\alpha N (\log \beta - 2T \|\varphi\|)} e^{4NTP(\text{Sing}, \varphi)},$$

which proves Proposition 8.5 with $Q = \log \beta - \log K - 2T \|\varphi\|$. \square

We now show how to sum over all the permitted possibilities for placing markers for surgery. The following consequence of Proposition 8.5 shows that we can recover the choice of J from any of the $v \in \mathcal{X}_J$.

Lemma 8.7. *If $J \neq J' \in \mathbb{J}_N^\alpha$ and $v \in \mathcal{X}_J$, $w \in \mathcal{X}_{J'}$, then there is $t \in [0, 4NT]$ such that $d(f_t v, f_t w) \geq \delta/5$.*

Proof. Since $J \neq J'$, there is $s \in \mathcal{A}$ such that $s \in J$ and $s \notin J'$. By (8.13), we have $d(f_t v, \text{Sing}) > 9\delta/5$ for some $t \in [s, s+T]$, while $d(f_t w, \text{Sing}) < 6\delta/5$ for every $t \in [s, s+T]$. Choosing the t that makes both inequalities true, we have $d(f_t v, f_t w) \geq 3\delta/5$. \square

It follows immediately that the set $F_N := \bigcup_{J \in \mathbb{J}_N^\alpha} \mathcal{X}_J$ is $(4NT, \delta/5)$ -separated. Moreover, (8.12) gives

$$(8.17) \quad \sum_{w \in F_N} e^{\int_0^{4NT} \varphi(f_t w) dt} \geq \binom{N-1}{\alpha N-1} e^{-\alpha N Q} e^{4NTP(\text{Sing}, \varphi)}.$$

Standard estimates on factorials and binomial coefficients give $\binom{N-1}{\alpha N-1} \geq e^{(-\alpha \log \alpha)N + o(N)}$ so that (8.17) gives

$$\Lambda(T^1 M, \varphi, \delta/5, 4NT) \geq e^{(-\alpha \log \alpha)N + o(N)} e^{-\alpha N Q} e^{4NTP(\text{Sing}, \varphi)}.$$

Taking a logarithm, dividing by $4NT$, and sending $N \rightarrow \infty$ gives

$$P(\varphi) \geq -\frac{\alpha}{4T} \log \alpha - \frac{\alpha Q}{4T} + P(\text{Sing}, \varphi).$$

For sufficiently small values of α , the right-hand side is greater than $P(\text{Sing}, \varphi)$, which completes the proof of Theorem B.

9. PROOF OF THEOREMS A, C AND D

Now we apply Theorem 3.1 to obtain Theorems A, C and D.

Proof of Theorem A. By Corollaries 7.5 and 7.8, if φ is Hölder continuous or $\varphi = q\varphi^u$, then it has the Bowen property on $\mathcal{G}(\eta)$ for all $\eta > 0$. Then Theorem 3.1 applies, yielding the statement of Theorem A. \square

Proof of Theorem C. For surfaces, we have $h_{\text{top}}(\text{Sing}) = 0$ and $\varphi^u(v) = 0$ for all $v \in \text{Sing}$, so $P(\text{Sing}, q\varphi^u) = 0$ for all $q \in \mathbb{R}$. We show that $P(q\varphi^u) > 0$ for all $q \in (-\infty, 1)$. The Liouville measure μ_L has

$0 < \int \varphi^u d\mu_L = -\int \lambda^+(\mu_L) = -h_{\mu_L}(\mathcal{F})$ by the Pesin entropy formula, so for every $q \in (-\infty, 1)$ we have

$$P(q\varphi^u) \geq h_{\mu_L}(\mathcal{F}) + \int q\varphi^u d\mu_L > h_{\mu_L} + \int \varphi^u d\mu_L = 0 = P(\text{Sing}, q\varphi^u).$$

Then Theorem A gives uniqueness and the desired properties for μ_q .

Since the flow is entropy expansive, the entropy map is upper semi-continuous, and so by work of Walters [23], the function $q \mapsto P(q\varphi^u)$ is C^1 on any interval where each $q\varphi^u$ has a unique equilibrium state. In particular, it is C^1 on $(-\infty, 1)$. \square

To show that the equilibrium states μ obtained in Theorem C are Bernoulli, we apply a result by Ledrappier, Lima, and Sarig [17] showing that if M is any 2-dimensional manifold, $\varphi: T^1M \rightarrow \mathbb{R}$ is Hölder or a scalar multiple of φ^u , and μ is a positive entropy ergodic equilibrium measure for the geodesic flow on T^1M , then μ is Bernoulli. Although their result is stated for positive entropy measures, this assumption is only used to guarantee that the measure has a positive Lyapunov exponent, see [18, Theorem 1.3]. Since our measure μ has $\mu(\text{Reg}) = 1$ and hence $\mu(\text{Sing}) = 0$, it has a positive exponent, so [17] applies.

We now prove Theorem D, and investigate the pressure gap for the potentials $q\varphi^u$ for higher dimensional manifolds.

Proof of Theorem D. For the proof of Theorem D, first observe that given any continuous φ , the set $\{q \in \mathbb{R} : P(\text{Sing}, q\varphi) < P(q\varphi)\}$ is open since both sides of the inequality vary continuously in q . Then Theorem D is a direct consequence of Theorems A and B. \square

As remarked after the statement of Theorem D, if M is a rank 1 manifold such that $h_{\text{top}}(\text{Sing}) = 0$, then we have $P(\text{Sing}, q\varphi^u) \leq 0$ for all $q \geq 0$ since $\varphi^u \leq 0$. Thus, the argument in the proof of Theorem C gives the pressure gap on $[0, 1)$. Since the gap is an open condition, it holds on $(-q_0, 1)$ for some $q_0 > 0$. Finally, we show that the pressure gap holds under a bounded range condition.

Lemma 9.1. *Let M be a closed rank 1 manifold and $\varphi: T^1M \rightarrow \mathbb{R}$ be continuous. If*

$$(9.1) \quad \sup_{v \in \text{Sing}} \varphi(v) - \inf_{v \in T^1M} \varphi(v) < h_{\text{top}}(\mathcal{F}) - h_{\text{top}}(\text{Sing}),$$

then $P(\text{Sing}, \varphi) < P(\varphi)$.

If $\dim(M) = 2$, then $h_{\text{top}}(\text{Sing}) = 0$, so the right hand side of (9.1) is just $h_{\text{top}}(\mathcal{F})$. If $\varphi = q\varphi^u$ or is Hölder, the bounded range hypotheses (9.1) gives another criterion which ensures that Theorem A applies. In

particular, it follows that the value of q_0 in Theorem D can be taken with $q_0 \geq (h_{\text{top}}(\mathcal{F}) - h_{\text{top}}(\text{Sing}))/2\|\varphi^u\|$.

Proof of Lemma 9.1. First rewrite (9.1) as

$$(9.2) \quad h_{\text{top}}(\text{Sing}) + \sup_{v \in \text{Sing}} \varphi(v) < h_{\text{top}}(\mathcal{F}) + \inf_{v \in T^1 M} \varphi(v).$$

The variational principle for $\mathcal{F}|_{\text{Sing}}$ gives

$$P(\text{Sing}, \varphi) = \sup_{\nu \in \mathcal{M}(\mathcal{F}|_{\text{Sing}})} \left\{ h_\nu(\mathcal{F}) + \int \varphi d\nu \right\} \leq h_{\text{top}}(\text{Sing}) + \sup_{v \in \text{Sing}} \varphi(v).$$

Now let m be the measure of maximal entropy for \mathcal{F} . Then

$$h_{\text{top}}(\mathcal{F}) + \inf \varphi = h_m(\mathcal{F}) + \inf \varphi \leq h_m(\mathcal{F}) + \int \varphi dm \leq P(\varphi).$$

Together with (9.2), these give $P(\text{Sing}, \varphi) < P(\varphi)$. \square

We note that Gromov's example [16, §6] can be modified to make $h_{\text{top}}(\mathcal{F}) - h_{\text{top}}(\text{Sing})$ arbitrarily small, so there is no hope that (9.1) yields a universal lower bound on q_0 . We do not know if a small entropy gap restricts the value of q_0 . Understanding this issue for the Gromov example would give insight into the general case.

10. EXAMPLES

In this section, we investigate examples of the geodesic flow on rank 1 manifolds with $\text{Sing} \neq \emptyset$. First, we give a class of manifolds for which we establish the existence of unique equilibrium states for a C^0 -generic set of potential functions. This class includes any rank 1 surface equipped with an analytic metric. The second example is a modification of an example due to Heintze in which we establish the uniqueness of an equilibrium state for $q\varphi^u$ for all $q \in \mathbb{R}$.

10.1. Examples where pressure gap holds generically. We show that when the singular set is a finite union of disjoint compact sets, on each of which the geodesic flow is uniquely ergodic, then the set of Hölder potentials for which there is a pressure gap is C^0 -generic.

Proposition 10.1. *Suppose that Sing is a union of disjoint compact sets Z_1, \dots, Z_k , on each of which the geodesic flow is uniquely ergodic. Let $H_0 \subset C(T^1 M)$ be the set of all ψ that are constant on a neighborhood of each Z_i , and let $H \subset C(T^1 M)$ be the set of all φ that are cohomologous to some $\psi \in H_0$. Then H is C^0 -dense in $C(T^1 M)$.*

Proof. Given $\varphi \in C(T^1M)$ and $T > 0$, consider the ergodic average function $\varphi_T(v) := \frac{1}{T} \int_0^T \varphi(f_s v) ds$. Then φ and φ_T are cohomologous; indeed, writing $\zeta(v) := \frac{1}{T} \int_0^T (T-s) \varphi(f_s v) dv$, an elementary computation shows that the derivative of ζ in the flow direction is $\varphi_T - \varphi$.

Let μ_i be the unique invariant measure on Z_i , and write $c_i = \int \varphi d\mu_i$. Given $\epsilon > 0$, there are T_1, \dots, T_k such that for every $T \geq T_i$, we have $|\varphi_T(v) - c_i| < \epsilon$ for every $v \in Z_i$. Let $T = \max_i T_i$, and let $\psi = \varphi_T$.

There is a function $\tilde{\psi} \in H_0$ taking the value c_i on a neighborhood of Z_i and having $\|\tilde{\psi} - \psi\|_{C^0} < \epsilon$. Let $\tilde{\varphi} = \tilde{\psi} + \varphi - \psi$, then $\tilde{\varphi}$ is cohomologous to $\tilde{\psi}$, so $\tilde{\varphi} \in H$, and we have $\|\tilde{\varphi} - \varphi\|_{C^0} = \|\tilde{\psi} - \psi\|_{C^0} < \epsilon$. \square

Under the hypotheses of Proposition 10.1, every $\varphi \in H$ is cohomologous to some $\psi \in H_0$ to which Theorem B applies, giving $P(\text{Sing}, \varphi) = P(\text{Sing}, \psi) < P(\psi) = P(\varphi)$. Since $P(\varphi)$ and $P(\text{Sing}, \varphi)$ vary continuously as φ varies (w.r.t. C^0), the set of potentials with the pressure gap is C^0 -open, and since it contains H , it is C^0 -dense. Writing C^h for the space of Hölder potentials on T^1M , observe that C^h is C^0 -dense in $C(T^1M)$, so the intersection $H \cap C^h$ is C^0 -dense in C^h . This shows that the set of Hölder potentials for which the pressure gap holds, which is clearly C^0 -open in the space of Hölder potentials, is C^0 -dense.

Analytic metrics on surfaces. For a rank 1 surfaces with an analytic metric, it is a folklore result that Sing is a finite union of periodic orbits; we sketch the idea of proof. If Sing is not a finite union of periodic orbits, then the geodesic flow has a transversal whose intersection with Sing is not discrete. In particular, there is a geodesic segment T and vectors $v_n, v \in \text{Sing}$, $v_n \neq v$, such that T is orthogonal to v , transverse to v_n , and $v_n \rightarrow v$. Locally, the geodesics γ_{v_n} intersect γ_v at most once, and so for almost all t close to 0, a short geodesic segment orthogonal to $f_t(v)$ contains a sequence of points x_n such that $x_n \rightarrow \pi g_t(v)$ and the Gaussian curvature of M at x_n is 0. Since curvature is real analytic, it must vanish along each of these geodesic segments and hence it is constant in a neighborhood of $\pi(v)$; since M is connected, it must vanish everywhere, which is a contradiction. Thus, Sing is a finite union of periodic orbits and so Proposition 10.1 applies.

Non-generic pressure gap and questions. The pressure gap does not hold generically if the manifold has a flat strip. One can take a potential φ supported near a periodic trajectory in the middle of the strip; if the support is small enough and the size of the potential large enough, one can guarantee that any regular trajectory has an ergodic average much smaller than the average along the periodic orbit, and

conclude that $P(\text{Sing}, \varphi) > P(\text{Reg}, \varphi)$. This inequality is stable under C^0 -perturbations of φ , so there is a C^0 -open set of potentials ψ for which $P(\text{Sing}, \psi) = P(\psi)$. It would be interesting to further investigate which classes of rank 1 manifolds have the pressure gap for C^0 -generic (or C^α -generic) Hölder potentials.

10.2. Heintze example. The following example of a rank 1 manifold is attributed to Heintze and described by Ballmann, Brin, and Eberlein [2, Example 4]. Consider an n -dimensional manifold N of constant negative curvature and finite volume with only one cusp. The cross section of the cusp is a flat $(n - 1)$ -dimensional torus T . Next cut off the cusp and flatten the manifold near the cut so the resulting manifold is locally isometric to the direct product of T and the unit interval. Now consider another copy of the same manifold and identify the two copies along T to obtain a manifold M with nonpositive sectional curvature. The rank of any tangent vector to a geodesic in T is n ; however, any tangent vector to a geodesic transverse to T has rank 1.

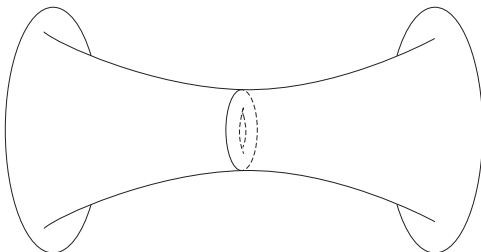


FIGURE 10.1. Modified Heintze's example

We now modify this example. We first assume for simplicity that $n = 3$ and the cross-section is a 2-torus T . Start by perturbing the metric on a compact subset of N so that there are two periodic orbits for which the corresponding invariant measures μ_1 and μ_2 have $\int \varphi^u d\mu_1 < \int \varphi^u d\mu_2$. Then choose T as above; by choosing T to lie far enough out in the cusp we can guarantee that curvature is still constant in a neighborhood of T . Instead of flattening the metric near the cut, replace the direct product metric on $T \times [0, 1]$ with a warped product, see for instance [20, p. 204] in which the tori are scaled by $\chi \cosh(d)$ where d is the distance from the center 2-torus and $\chi > 0$. The fact that $\cosh(d)$ has a minimum at $d = 0$ means that the central 2-torus is totally geodesic; all of the vectors tangent to it are singular and have exponent zero in the direction tangent to the 2-torus. However now the sectional curvature in the direction orthogonal to the central 2-torus is negative and gives a nonzero Lyapunov exponent, see Figure 10.1.

For this modified example, Sing consists of vectors tangent to the center 2-torus. Every $v \in \text{Sing}$ has zero Lyapunov exponent in the center direction, but the other Lyapunov exponents are nonzero since the sectional curvature corresponding to these directions is nonzero. By the warped product construction every vector in Sing will have the same positive Lyapunov exponent $\lambda > 0$, and since $h(\text{Sing}) = 0$, we know $P(\text{Sing}, q\varphi^u) = -q\lambda$ for all $q \in \mathbb{R}$. By varying the parameter χ in the construction, we can vary λ so that $\int \varphi^u d\mu_1 < -\lambda < \int \varphi^u d\mu_2$. Thus for all $q > 0$ we have

$$P(q\varphi^u) \geq q \int \varphi^u d\mu_2 > -q\lambda = P(\text{Sing}, q\varphi^u),$$

and the corresponding inequality for $q < 0$ follows by considering μ_1 .

Finally, since $h_{\text{top}}(\mathcal{F}) = P(0) > 0$ we see that $P(q\varphi^u) > P(\text{Sing}, q\varphi^u)$ for all $q \in \mathbb{R}$. Thus we have an example of a compact smooth 3-manifold M that is rank 1 of nonpositive curvature for which $\text{Sing} \neq \emptyset$, and indeed M does not support a metric of strictly negative curvature (since $\pi_1(M)$ contains \mathbb{Z}^2), but on the other hand $q\varphi^u$ has a unique equilibrium state for every $q \in \mathbb{R}$, which is fully supported. In particular, the Liouville measure is the unique equilibrium state for φ^u .

Acknowledgments. Much of this work was carried out in a SQuaRE program at the American Institute of Mathematics. We thank AIM for their support and hospitality.

REFERENCES

1. Werner Ballmann, *Lectures on spaces of nonpositive curvature*, DMV Seminar, vol. 25, Birkhäuser Verlag, Basel, 1995, With an appendix by Misha Brin.
2. Werner Ballmann, Misha Brin, and Patrick Eberlein, *Structure of manifolds of nonpositive curvature. I*, Ann. of Math. (2) **122** (1985), no. 1, 171–203.
3. Rufus Bowen, *Some systems with unique equilibrium states*, Math. Systems Theory **8** (1974/75), no. 3, 193–202.
4. Rufus Bowen and David Ruelle, *The ergodic theory of Axiom A flows*, Invent. Math. **29** (1975), no. 3, 181–202.
5. Keith Burns and Katrin Gelfert, *Lyapunov spectrum for geodesic flows of rank 1 surfaces*, Discrete Contin. Dyn. Syst. **34** (2014), no. 5, 1841–1872.
6. Vaughn Climenhaga and Daniel J. Thompson, *Unique equilibrium states for flows and homeomorphisms with non-uniform structure*, Adv. Math. **303** (2016), 745–799.
7. W. A. Coppel, *Disconjugacy*, Lecture Notes in Mathematics, Vol. 220, Springer-Verlag, Berlin-New York, 1971.
8. Patrick Eberlein, *Geodesic flows on negatively curved manifolds. II*, Trans. Amer. Math. Soc. **178** (1973), 57–82.

9. ———, *Geodesic flows in manifolds of nonpositive curvature*, Smooth ergodic theory and its applications (Seattle, WA, 1999), Proc. Sympos. Pure Math., vol. 69, Amer. Math. Soc., Providence, RI, 2001, pp. 525–571.
10. Patrick B. Eberlein, *Geometry of nonpositively curved manifolds*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1996.
11. Ernesto Franco, *Flows with unique equilibrium states*, Amer. J. Math. **99** (1977), no. 3, 486–514.
12. Katrin Gelfert and Rafael O. Ruggiero, *Geodesic flows modeled by expansive flows*, (2017), Preprint, arXiv:1703.07455.
13. Katrin Gelfert and Barbara Schapira, *Pressures for geodesic flows of rank one manifolds*, Nonlinearity **27** (2014), no. 7, 1575–1594.
14. Marlies Gerber and Amie Wilkinson, *Hölder regularity of horocycle foliations*, J. Differential Geom. **52** (1999), no. 1, 41–72.
15. Anatole Katok and Boris Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, Cambridge, 1995.
16. Gerhard Knieper, *The uniqueness of the measure of maximal entropy for geodesic flows on rank 1 manifolds*, Ann. of Math. (2) **148** (1998), no. 1, 291–314.
17. François Ledrappier, Yuri Lima, and Omri Sarig, *Ergodic properties of equilibrium measures for smooth three dimensional flows*, Comment. Math. Helv. **91** (2016), no. 1, 65–106.
18. Yuri Lima and Omri Sarig, *Symbolic dynamics for three dimensional flows with positive topological entropy*, J. Eur. Math. Soc. (2017), To appear, arXiv:1408.3427.
19. Anthony Manning, *Curvature bounds for the entropy of the geodesic flow on a surface*, J. London Math. Soc. (2) **24** (1981), no. 2, 351–357.
20. Barrett O’Neill, *Semi-Riemannian geometry*, Pure and Applied Mathematics, vol. 103, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983, With applications to relativity.
21. William Parry, *Equilibrium states and weighted uniform distribution of closed orbits*, Dynamical systems (College Park, MD, 1986–87), Lecture Notes in Math., vol. 1342, Springer, Berlin, 1988, pp. 617–625.
22. Peter Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, vol. 79, Springer-Verlag, New York-Berlin, 1982.
23. ———, *Differentiability properties of the pressure of a continuous transformation on a compact metric space*, J. London Math. Soc. (2) **46** (1992), no. 3, 471–481.

K. BURNS, DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, IL 60208, *E-mail address*: burns@math.northwestern.edu

V. CLIMENHAGA, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON, HOUSTON, TX 77204, *E-mail address*: climenha@math.uh.edu

T. FISHER, DEPARTMENT OF MATHEMATICS, BRIGHAM YOUNG UNIVERSITY, PROVO, UT 84602, *E-mail address*: tfisher@mathematics.byu.edu

D. J. THOMPSON, DEPARTMENT OF MATHEMATICS, THE OHIO STATE UNIVERSITY, COLUMBUS, OH 43210, *E-mail address*: thompson@math.osu.edu